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Decomposition and construction of higher-dimensional neighbourhood operations

Atsushi Imiya[1,*]

Institute of Management and Information Technologies, Chiba University, Yayoi-cho 1-33, Inage-ku, Chiba 263-8522, Japan

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ABSTRACT

We prove that the $2n$-neighbourhood in an n-dimensional digital space is decomposed into the $2(n-1)$-neighbourhoods in the mutually orthogonal $(n-1)$-dimensional digital spaces. This decomposition and construction relation of the neighbourhoods and objects implies that morphological operations in an n-dimensional digital space can be computed as the union of one- and two-dimensional morphological operations on isothetic digital lines and planes intersecting with the digital object in the digital space.

1. Introduction

This paper develops a method to construct higher-dimensional digital morphological operations from a collection of one- and two-dimensional set operations along digital isothetic lines and on digital planes, respectively, in a space. Together with decomposition of digital objects, the decomposition of the neighbourhood shows that the neighbourhood-based operation in the higher-dimensional digital spaces can be decomposed into the union of neighbourhood-based operations in the lower-dimensional digital spaces [1].

Decompositions of morphological operations, such as closing, opening, the hit-or-miss transform, distance transform, boundary detection, skeletonisation [2–4] and thinning, clarify that the higher-dimensional operations are hierarchically constructed from those in the lower-dimensional spaces. Mathematically reformulations of algorithms based on decomposition properties of morphological operations bring theoretical bridge between mathematical descriptions and programme developments of morphological operations [5,6].

2. Mathematical preliminaries

Setting $\mathbf{R}^n$ to be an n-dimensional Euclidean space, we express vectors in $\mathbf{R}^n$ as $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^\top$. Let $\mathbf{Z}$ be the set of all integers. The n-dimensional digital space $\mathbf{Z}^n$ is set of all $\boldsymbol{x}$ for which all x_i are integers. Then, we define the voxel centred at points in $\mathbf{Z}^n$ as a unit hypercube in $\mathbf{R}^n$.

Definition 1. The voxels centred at the point $\boldsymbol{y} \in \mathbf{Z}^n$ in $\mathbf{R}^n$ is

$$\mathbf{V}(\boldsymbol{y}) = \left\{\boldsymbol{x} \,|\, |\boldsymbol{x} - \boldsymbol{y}|_\infty \leq \frac{1}{2}\right\}. \quad (1)$$

In this paper, we deal with the connectivity and adjacency of the centroids of the voxels [7], which are elements of $\mathbf{Z}^n$.

The sets $\mathbf{F}\oplus\mathbf{G}$ and $\mathbf{F}\ominus\mathbf{G}$ such that

$$\mathbf{F} \oplus \mathbf{G} = \bigcup_{\boldsymbol{y}\in\mathbf{G}} \left(\bigcup_{\boldsymbol{x}\in\mathbf{F}} \{\boldsymbol{x} + \boldsymbol{y}\right), \qquad \mathbf{F} \ominus \mathbf{G} = \bigcap_{\boldsymbol{y}\in\mathbf{G}} \left(\bigcup_{\boldsymbol{x}\in\mathbf{F}} \{\boldsymbol{x} + \boldsymbol{y}\}\right) \quad (2)$$

are called the Minkowski addition (dilation) and Minkowski subtraction (erosion) of $\mathbf{F}$ and $\mathbf{G}$, respectively. The translation of $\mathbf{F}$ by

* Corresponding author.
E-mail address: imiya@faculty.chiba-u.jp
[1] IAPR Fellow

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$\boldsymbol{a} \in \mathbf{Z}^n$ is $\mathbf{F}(\boldsymbol{a}) = \bigcup_{\boldsymbol{x}\in\mathbf{F}}\{\boldsymbol{x}+\boldsymbol{a}\} = \mathbf{F}\oplus\{\boldsymbol{a}\}$. The morphological operations and set operations fulfil the following relations [8].

Theorem 1. *The Minkowski addition (dilation) and Minkowski subtraction(erosion),*

$$\mathbf{F}\ominus\mathbf{G} = \overline{\mathbf{F}\oplus\overline{\mathbf{G}}}, \tag{3}$$

$$\mathbf{F}\oplus(\mathbf{G}\cup\mathbf{H}) = (\mathbf{F}\oplus\mathbf{G})\cup(\mathbf{F}\oplus\mathbf{H}), \tag{4}$$

$$\mathbf{F}\ominus(\mathbf{G}\cup\mathbf{H}) = (\mathbf{F}\ominus\mathbf{G})\cap(\mathbf{F}\ominus\mathbf{H}) \tag{5}$$

are satisfied.

The theorem derives the next corollary among morphological operations and set operations.

Corollary 1. *If* $\mathbf{F}\cap\mathbf{G} = \emptyset$, *the equalities*

$$(\mathbf{F}\cup\mathbf{G})\oplus\mathbf{H} = (\mathbf{F}\oplus\mathbf{H})\cup(\mathbf{G}\oplus\mathbf{H}), \tag{6}$$

$$(\mathbf{F}\cup\mathbf{G})\ominus\mathbf{H} = (\mathbf{F}\ominus\mathbf{H})\cup(\mathbf{G}\ominus\mathbf{H}) \tag{7}$$

are satisfied.

Proof. Since Eq. (6) is straightforward from Eq. (4), we prove Eq. (7).

$$\begin{aligned}(\mathbf{F}\cup\mathbf{G})\ominus\mathbf{H} &= \bigcap_{\boldsymbol{x}\in\mathbf{H}}(\mathbf{F}\cup\mathbf{G})(\boldsymbol{x})\\ &= \{\boldsymbol{x}+\boldsymbol{y}\,|\,\forall\boldsymbol{x}\in\mathbf{H},\ \exists\boldsymbol{y}\in(\mathbf{F}\cup\mathbf{G})\}\\ &= \{\boldsymbol{x}+\boldsymbol{y}\,|\,\forall\boldsymbol{x}\in\mathbf{H},\ \exists\boldsymbol{y}\in\mathbf{F}\}\\ &\quad\cup\{\boldsymbol{x}+\boldsymbol{y}\,|\,\forall\boldsymbol{x}\in\mathbf{H},\ \exists\boldsymbol{y}\in\mathbf{G}\}\\ &= (\mathbf{F}\ominus\mathbf{H})\cup(\mathbf{G}\ominus\mathbf{H}).\end{aligned}$$

□

Next, we define the multidirectional multislice decomposition of a digital point $\mathbf{F}\subset\mathbf{Z}^n$.

Definition 2. For $\mathbf{F}\subset\mathbf{Z}^n$, digital linear subspace is

$$\mathbf{F}_k = \{\boldsymbol{x}\,|\,\boldsymbol{x}\in\mathbf{F}, x_k = 0\} \tag{8}$$

and the digital linear manifold is

$$\mathbf{F}_{k\alpha} = \{\boldsymbol{x}\,|\,\boldsymbol{x}\in\mathbf{F}_k\oplus\{\alpha\boldsymbol{e}_k\}, \alpha\in\mathbf{Z}\}. \tag{9}$$

For $\alpha_+(k) = \max_{\mathbf{F}\cap\mathbf{F}_{k\alpha}\neq\emptyset}\alpha$ and $\alpha_-(k) = \min_{\mathbf{F}\cap\mathbf{F}_{k\alpha}\neq\emptyset}\alpha$, setting $\mathcal{N}(k) = \{\alpha|\alpha_-(k)\leq\alpha\leq\alpha_+(k)\}$,$\mathbf{F}_{k\alpha}$ satisfies the following property on the multidirectional multislice decomposition.

Property 1. *A point set* $\mathbf{F}\subset\mathbf{Z}^n$ *is decomposed as* $\mathbf{F} = \cup_{k=1}^n\left(\cup_{\alpha\in\mathcal{N}(k)}\mathbf{F}_{k\alpha}\right)$.

This property derives the following lemma.

Lemma 1. *The hierarchical decomposition of the point sets is*

$$\mathbf{F}_{k(l)\alpha(l)} = \bigcup_{k(l)=1}^{n-l}\left(\bigcup_{\alpha(l)\in\mathcal{N}(k(l))}\mathbf{F}_{k(l)\alpha(l)}\right), \tag{10}$$

for $l = 0, 1, 2, \cdots, n-1$.

Proof. The relations

$$\mathbf{F}_{k\alpha} = \bigcup_{l=1}^{n-1}\left(\bigcup_{\beta\in N(l)}\mathbf{F}_{k\alpha\,l\beta}\right), \mathbf{F}_{k\alpha\,l\beta} = \bigcup_{m=1}^{n-2}\left(\bigcup_{\gamma\in N(m)}\mathbf{F}_{k\alpha\,l\beta\,m\gamma}\right) \tag{11}$$

are satisfied. Hierarchical application of these relations leads Eq. (10). □

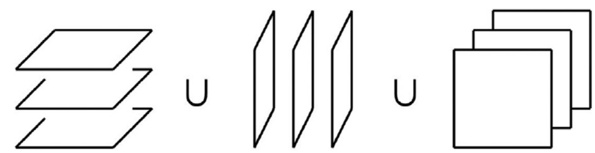

Fig. 1. Digital point set and its decomposition. The multidirectional multislice decomposition of a digital point set in a three-dimensional digital space is shown.

Fig. 1 shows the multidirectional multislice decomposition of a digital point set in a three-dimensional digital space.

For

$$\mathbf{F} = \bigcup_{k=1}^n\bigcup_{\alpha\in\mathcal{N}(k)}\mathbf{F}_{k\alpha},\ \ \mathbf{G} = \bigcup_{k=1}^n\bigcup_{\beta\in\mathcal{N}(k)}\mathbf{G}_{l\beta}, \tag{12}$$

the Minkowski addition (dilation) and Minkowski subtraction (erosion) are

$$\mathbf{F}\oplus\mathbf{G} = \left(\bigcup_{k=1}^n\bigcup_{\alpha\in\mathcal{N}(k)}\mathbf{F}_{k\alpha}\right)\oplus\left(\bigcup_{k=1}^n\bigcup_{\beta\in\mathcal{N}(k)}\mathbf{G}_{k\beta}\right), \tag{13}$$

$$\mathbf{F}\ominus\mathbf{G} = \left(\bigcup_{k=1}^n\bigcup_{\alpha\in\mathcal{N}(k)}\mathbf{F}_{k\alpha}\right)\ominus\left(\bigcup_{k=1}^n\bigcup_{\beta\in\mathcal{N}(k)}\mathbf{G}_{k\beta}\right). \tag{14}$$

Using Eqs. (6) and (7), Eqs. (13) and (14) derive the following theorem.

Theorem 2. *The Minkowski addition (dilation) and Minkowski subtraction (erosion) are computed as*

$$\mathbf{F}\oplus\mathbf{G} = \bigcup_{k=1}^n\bigcup_{\gamma\in N_{\alpha\beta}(k)}\mathbf{F}_{k\gamma}\oplus\mathbf{G}_{k\gamma}, \tag{15}$$

$$\mathbf{F}\ominus\mathbf{G} = \bigcup_{k=1}^n\bigcup_{\gamma\in N_{\alpha\beta}(k)}\mathbf{F}_{k\gamma}\ominus\mathbf{G}_{k\gamma}, \tag{16}$$

where $\mathcal{N}_{\alpha\beta}(k) = \{\gamma\,|\min(\alpha_-,\beta_-)\leq\gamma\leq\max(\alpha_+,\beta_+)\}$.

Since the opening and closing are $(\mathbf{F}\ominus\mathbf{G})\oplus\mathbf{H}$ and $(\mathbf{F}\oplus\mathbf{G})\ominus\mathbf{H}$, respectively, theorem 2 implies the following corollary.

Corollary 2. *The opening and closing satisfy the relations*

$$(\mathbf{F}\ominus\mathbf{G})\oplus\mathbf{H} = \bigcup_{k=1}^n\bigcup_{\delta\in\mathcal{N}_{\alpha\beta\gamma}(k)}(\mathbf{F}_{k\delta}\ominus\mathbf{G}_{k\delta})\oplus\mathbf{H}_{k\delta}, \tag{17}$$

$$(\mathbf{F}\oplus\mathbf{G})\ominus\mathbf{H} = \bigcup_{k=1}^n\bigcup_{\delta\in N_{\alpha\beta\gamma}(k)}(\mathbf{F}_{k\delta}\oplus\mathbf{G}_{k\delta})\ominus\mathbf{H}_{k\delta}, \tag{18}$$

for $\mathcal{N}_{\alpha\beta\gamma}(k) = \{\delta|\min(\alpha_-,\beta_-,\gamma_-)\leq\delta\leq\max(\alpha_+,\beta_+,\gamma_+)\}$.

Theorem 2 and Corollary 2 imply that all the Minkowski addition (dilation), the Minkowski subtraction (erosion), opening and closing can be constructed from lower-dimensional ones.

Using embedding operation defined bellow, we claify geometry property of voxel union.

Definition 3. The embedding of a point set $\mathbf{F}\in\mathbf{Z}^n$ into $\mathbf{R}^n$ is $\mathcal{F} = \cup_{\boldsymbol{x}\in\mathbf{F}}\mathbf{V}(\boldsymbol{x})$.

$\mathcal{F}$ is a union of voxels.

We define the dual grid

Definition 4. The dual grid [9,10] of $\mathbf{Z}^n$ is $\mathbf{D}^n = \mathbf{Z}^n\oplus\left\{\frac{1}{2}\boldsymbol{e}\right\}$, where $\boldsymbol{e} = \sum_{i=1}^n\boldsymbol{e}_i$ for $\boldsymbol{e}_i = (\overbrace{0,0,\cdots,0}^{i-1},1,0,0,\cdots,0)^\top$.

The embedded point set possesses the next property.

Property 2. *The polytope $\mathcal{F}$ is an isothetic Nef-polytope [11,12,13], which is a union of voxels connected by the faces of voxels. The vertices of $\mathcal{F}$ lie on the dual grid $\mathbf{D}^n$.*

Since a sub-grid point $\boldsymbol{p}$ in the unit hypercube $[0, 1]^n$ is expressed as $\boldsymbol{p} = \sum_{i=1}^{n} \frac{\alpha(i)}{k}\boldsymbol{e}_i$, for $\alpha(i) = 0, 1, 2, \cdots, n-1$, where k is an appropriate positive integer, we have the following definition.

Definition 5. The k-sub-grid is

$$\mathbf{Z}_k^n = \left\{ \boldsymbol{y} | \boldsymbol{y} = \boldsymbol{x} + \sum_{i=1}^{n} \frac{\alpha(i)}{k}\boldsymbol{e}_i,\ \boldsymbol{x} \in \mathbf{Z}^n \right\}. \tag{19}$$

Using sub-grids, resampling of polytope is achieved.

Definition 6. The resampling of $\mathcal{F} \in \mathbf{R}^n$ in the k-sub-grid $\mathbf{Z}_k^n$ is expressed as $\mathbf{F}^k$.

3. Neighbourhood operations

The $2n$-neighbourhood of the origin in $\mathbf{Z}^n$ is

$$\mathbf{N}^n = \{\boldsymbol{x} \,|\, |x_i| = 1,\ \boldsymbol{x} = (x_1, x_2, \cdots, x_n)^\top\}. \tag{20}$$

Let $\mathbf{N}(\boldsymbol{x}) = \mathbf{N}^n \oplus \{\boldsymbol{x}\}$ for $\boldsymbol{x} \in \mathbf{Z}^n$. Using the neighbourhood, connectivity of a pair of points a path between a pair of points in point set connectivity of a pair of in a point set are defined.

Definition 7. If $\boldsymbol{y} \in \mathbf{N}(\boldsymbol{x})$ and $\boldsymbol{x} \in \mathbf{N}(\boldsymbol{y})$, $\boldsymbol{x}$ and $\boldsymbol{y}$ are connected to each other.

Definition 8. For $\boldsymbol{y} \notin \mathbf{N}(\boldsymbol{x})$, if there exists at least one sequence $\boldsymbol{p}_{i+1} \in \mathbf{N}(\boldsymbol{p}_i)$ and $\boldsymbol{p}_i \in \mathbf{N}(\boldsymbol{p}_{i+1})$ for $i = 1, 2, \cdots k-1$, the string $\{\boldsymbol{p}\}_{i=1}^{k}$ is a path from $\boldsymbol{p}_1 := \boldsymbol{x}$ to $\boldsymbol{p}_2 := \boldsymbol{y}$.

Definition 9. For a pair of points $\boldsymbol{x}$ and $\boldsymbol{y}$, if there exists a path between them, this pair is connected.

Existence of paths defines connected components.

Definition 10. For $\mathbf{F} \in \mathbf{Z}^n$, if there exist at least a path between any pairs of points in **F, F** is a connected component.

On the digital line $\mathbf{Z}$, the neighbourhood $\mathbf{N}^1$ of the point 0 is $\mathbf{N}^1 = \{-1, 1\}$ and a digital object is a string of points $\mathbf{O} = \{k\}_{k=n}^{m}$ for $m > n$ and $m, n \in \mathbf{Z}$. The dilation and erosion of a collection of points are concatenation and elimination of points to both endpoints of a string, respectively.

From the linear neighbourhood in $\mathbf{Z}^n$ such that

$$\mathbf{N}_k^1 = \{\boldsymbol{x} \,|\, |x_k| = 1,\ x_i = 0,\ i \neq k\}, \tag{21}$$

we can construct $\mathbf{N}^n$ as

$$\mathbf{N}^n = \bigcup_{k=1}^{n} \mathbf{N}_k^1, \tag{22}$$

$$\mathbf{N}^n = \bigcup_{k=1}^{n} \mathbf{N}_k^{n-1}, \mathbf{N}_k^{n-1} = \mathbf{N}^n \setminus \mathbf{N}_k^1, \tag{23}$$

$$\mathbf{N}_k^{n-1} = \bigcup_{l=1}^{n-1} \mathbf{N}_{kl}^{n-2}, \mathbf{N}_{kl}^{n-2} = \mathbf{N}_k^{n-1} \setminus \mathbf{N}_l^1, \tag{24}$$

$$\mathbf{N}_{kl}^{n-2} = \bigcup_{m=1}^{n-2} \mathbf{N}_{klm}^{n-3}, \mathbf{N}_{klm}^{n-3} = \mathbf{N}_{kl}^{n-2} \setminus \mathbf{N}_m^1. \tag{25}$$

Equations (23), (24) and (25) imply the following property.

Property 3. *A neighbourhood in a higher-dimensional digital space can be decomposed into the union of neighbourhoods in lower-dimensional digital spaces.*

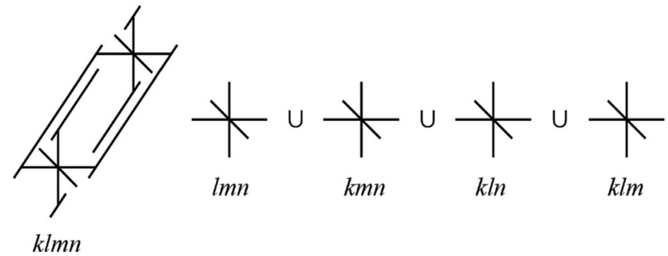

Fig. 2. Decomposition of a neighbourhood. The 8-neighbourhood in a four-dimensional digital space is decomposed into four mutually orthogonal 6-neighbourhoods in the three-dimensional digital spaces.

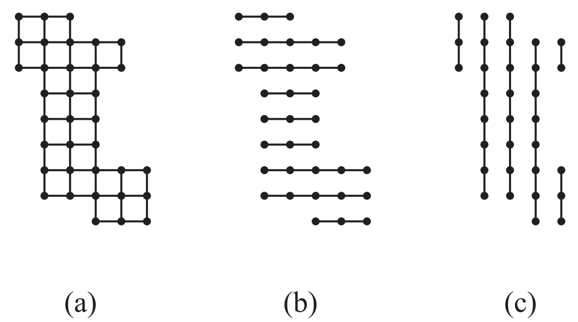

Fig. 3. One-dimensional operations for a two-dimensional object. (a) Four-connected object on the digital plane. (b) Neighbourhood operations on the horizontal isothetic lines on the digital plane. (c) Neighbourhood operations on the vertical isothetic lines on the digital plane.

These recursive decompositions of the neighbourhoods derive the following theorem.

Theorem 3. *The $2n$-neighbourhood in $\mathbf{Z}^n$ is decomposed as*

$$\mathbf{N}_{k(1)k(2)\cdots k(l)}^{n-l} = \bigcup_{k(l)=1}^{n-l} \mathbf{N}_{k(1)k(2)\cdots k(l+1)}^{n-(l+1)},$$
$$\mathbf{N}_{k(1)k(2)\cdots k(l+1)}^{n-(l+1)} = \mathbf{N}_{k(1)k(2)\cdots k(l)}^{n-l} \setminus \mathbf{N}_{k(l+1)}^1, \tag{26}$$

for $l = 0, 1, 2, \cdots n-1$.

Fig. 2 shows that the 8-neighbourhood in a four-dimensional digital space is decomposed into four mutually orthogonal 6-neighbourhoods in the three-dimensional digital spaces.

Fig. 3 illustrates the neighbourhood operations on the horizontal and vertical isothetic lines on a digital plane. The connectivity of the four connected object shown in (a) is computed by using the connectivity on the horizontal and the vertical isothetic lines on the digital plane.

4. Digital simplex, complex and object

We first define digital simplices in $\mathbf{Z}^n$ assuming $2n$-connectivity of points.

Definition 11. A digital k-simplex in $\mathbf{Z}^n$ is the union of vertices of unit k-cube in $\mathbf{Z}^n$ for $1 \leq k \leq n$ assuming $2n$-connectivity of points.

The following procedure constructs digital simplices.

Lemma 2. *The recursive form*

$$\mathbf{S}(0; n) = \{\boldsymbol{s}_0^n | \boldsymbol{s}_0^n = \{\boldsymbol{0}\}\},$$
$$\mathbf{S}(k; n) = U\left(\{\mathbf{s}_k^n | \mathbf{s}_k^n = \mathbf{s}_{k-1}^n \cup (\mathbf{s}_{k-1}^n \oplus \mathbf{e}_i),\ \mathbf{e}_i \notin \mathbf{s}_{k-1}^n\}\right) \tag{27}$$

constructs digital k-simplex $\mathbf{s}_k^n$ containing the origin $\boldsymbol{0}$ for $k \geq 1$, where the operation $U(\{\cdot\})$ removes redundant elements in the set $\{\cdot\}$.

Proof. The elements of $\mathbf{S}(1;n) = \{\{0, \boldsymbol{e}_1\}, \{0, \boldsymbol{e}_2\}, \cdots, \{0, \boldsymbol{e}_n\}\}$ are all digital 1-simplices containing the origin. Let $\mathbf{S}(k-1;0)$ to be the collection of all digital $(k-1)$-simplices containing the origin. All of $\mathbf{s}^n_{k-1} \cup (\mathbf{s}^n_{k-1} \oplus \mathbf{e}_i)$ for $\boldsymbol{e}_i \notin \mathbf{s}^n_{k-1}$ and $\mathbf{s}^n_{k-1} \in \mathbf{S}(k-1;n)$ are digital k-simplices containing the origin in $\mathbf{Z}^n$. □

Definition 12. For $0 \le k \le n$, we call k of $\mathbf{s}^n_k$ the dimension of the simplices.

All digital simplex in $\mathbf{Z}^n$ is $\mathbf{s}^n_k(\boldsymbol{p}) = \mathbf{s}^n_k \oplus \{\boldsymbol{p}\}$ for $\boldsymbol{p} \in \mathbf{Z}^n$. Digital complices are defined through digital simplices.

Definition 13. A connected digital complex is constructed as a union of connected digital simplices.

The following lemma shows geometric properties of connected digital simplices.

Lemma 3. *The common set of a connecting pair of digital k-simplices is a digital l-simplex for* $0 \le l \le k-1$.

Proof. For $\boldsymbol{q}_k = \sum_{i=1}^{k-l} \boldsymbol{e}_i$, $\mathbf{s}^n_k$ and $\mathbf{s}^n_k \oplus \{\boldsymbol{q}_k\}$ is connected. Since $\mathbf{s}^n_k \cap (\mathbf{s}^n_k \oplus \{\boldsymbol{q}_k\}) = \mathbf{s}^n_l$, the common set of two connecting digital simplices is a collection of digital simplices. □

Since a digital 4-simplex in $\mathbf{Z}^4$ is the collection of all vertices of the unit 4-cube containing the origin, a digital complex is

$$\begin{aligned}\mathbf{C} &= \mathbf{s}^4_4 \cup \mathbf{s}^4_4(\boldsymbol{e}_1) \\ &= \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_{12}, \boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{23}, \boldsymbol{e}_{24}, \boldsymbol{e}_{34}, \\ &\quad \boldsymbol{e}_{123}, \boldsymbol{e}_{124}, \boldsymbol{e}_{134}, \boldsymbol{e}_{234}, \boldsymbol{e}_{1234}, \\ &\quad \boldsymbol{e}_{1^2 2}, \boldsymbol{e}_{1^2 3}, \boldsymbol{e}_{1^2 4}, \boldsymbol{e}_{1^2 23}, \boldsymbol{e}_{1^2 24}, \boldsymbol{e}_{1^2 34}, \boldsymbol{e}_{1^2 234}\},\end{aligned} \tag{28}$$

where $\boldsymbol{e}_{\alpha\beta\cdots\gamma\delta} = \boldsymbol{e}_{\alpha\beta\cdots\gamma} + \boldsymbol{e}_\delta$, that is, $\boldsymbol{e}_{12\cdots n} = \boldsymbol{e}$ and $\boldsymbol{e}_{\alpha\alpha} = \boldsymbol{e}_{\alpha^2} = 2\boldsymbol{e}_\alpha$. Furthermore, the interface of digital simplices in this digital complex is

$$\mathbf{f} = \mathbf{s}^4_4 \cap \mathbf{s}^4_4(\boldsymbol{e}_1) = \{\boldsymbol{e}_1, \boldsymbol{e}_{12}, \boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{123}, \boldsymbol{e}_{124}, \boldsymbol{e}_{134}, \boldsymbol{e}_{1234}\}. \tag{29}$$

Dimensions of the simplices in a digital complex define geometrical hierarchy of connected components.

Definition 14. We call a point set $\mathbf{P} \subset \mathbf{Z}^n$ a digital k-complex if there exists a point set $\mathbf{P}_k \subset \mathbf{P}$ such that $\mathbf{P} = \cup_{\boldsymbol{p} \in \mathbf{P}_k} \boldsymbol{s}^n_k(\boldsymbol{p})$ and does not exist $\mathbf{P}_{k+1} \subset \mathbf{P}$ such that $\mathbf{P} = \cup_{\boldsymbol{p} \in \mathbf{P}_{k+1}} \boldsymbol{s}^n_{k+1}(\boldsymbol{p})$.

Definition 15. For a digital k-complex $\mathbf{P}$, let

$$\mathbf{Q} = \arg\max_{\mathbf{A}} \left\{ \mathrm{cad}\left(\bigcup_{\boldsymbol{q} \in \mathbf{A} \subset \mathbf{P}} \mathbf{s}^n_{k+1}(\boldsymbol{q}) \right) \right\}. \tag{30}$$

We call $\mathbf{R} = \mathbf{P} \setminus \mathbf{Q}$ the k-wall, since $\mathbf{R}$ is a union of connecting digital k-simplices.

These definitions imply that an object contains digital k-simplices for $k \le n-1$ as the connected components [1,14,15].

These geometric properties of digital simplices and complices define thickness (or width) of digital objects.

Definition 16. The thick digital n-complex is a union of digital simplices connected by digital $(n-1)$-simplices.

The thickness of a digital complex defines the digital object.

Definition 17. If the number of connected digital simplices in a thick digital n-complex $\mathbf{F}$ is finite and if the complement of $\mathbf{F}$ is a thick n-complex, we call $\mathbf{F}$ a digital object.

Furthermore, we additionally define a thin digital object.

Definition 18. We call a connected component of digital k-simplices for $k \le (n-1)$ a thin digital object.

Therefore, we have the next property.

Property 4. *The minimum thickness of a thin digital object is one.*

On $\mathbf{Z}$, a digital object is a finite union of finite intervals

$$\mathbf{I} = \bigcup_{i=1}^{n} \mathbf{I}_i, \ \mathbf{I}_i = [a_i, b_i], \tag{31}$$

where a quadplet of integers a_i, a_{i+1}, b_i and b_{i+1} such that $a_i < a_{i+1}$ and $b_i < b_{i+1}$ with the condition $(a_{i+1} - b_i) \ge 3$.

The objects in $\mathbf{R}^n$ is defined using union of voxels, whose centres lie in a digital n-complices.

Definition 19. For a thin object $\mathbf{T}$ in $\mathbf{Z}^n$, we call the embedding of $\mathbf{T}$ in $\mathbf{R}^n \mathcal{T} = \cup_{\boldsymbol{x} \in \mathbf{T}} \mathbf{V}(\boldsymbol{x})$ an imperfect voxel object.

Definition 20. In $\mathbf{R}^n$, if the complement of voxel object $\mathcal{P}$ is an imperfect voxel object, we call $\mathcal{P}$ a perfect voxel object.

For the point set $\mathbf{P} = \{\boldsymbol{x} | \sum_{i=1}^n |x_i| \le k, \ k \ge 1\}$ the object $\mathcal{P} = \bigcup_{\boldsymbol{x} \in \mathbf{P}} V(\boldsymbol{x})$ is an imperfect object, since the thicknesses of $\mathcal{V}_{\pm ki} = V(\pm k\boldsymbol{e}_i)$, $i = 1, 2, \cdots, n$ are one. For the point set $\mathbf{Q} = \{\boldsymbol{x} | |x_i| \le k, k \ge 1\}$ the object $\mathcal{Q} = \bigcup_{\boldsymbol{x} \in \mathbf{Q}} V(\boldsymbol{x})$ is a perfect object since the minimum thickness of $\mathbf{Q}$ is $2k+1$.

5. Digital boundary manifold

We define the boundary of a point set in $\mathbf{Z}^n$.

Definition 21. For a point set $\mathbf{F}$, we call $\partial_- \mathbf{F} = \mathbf{F} \setminus (\mathbf{F} \ominus \mathbf{N}^n)$ and $\partial_+ \mathbf{F} = (\mathbf{F} \oplus \mathbf{N}^n) \setminus \mathbf{F}$ the internal and external boundaries of $\mathbf{F}$, respectively.

For the internal and external boundaries, we have the following relations.

Lemma 4.

$$\mathbf{F} \setminus (\mathbf{F} \ominus \mathbf{N}^n) = \bigcup_{k=1}^{n} \bigcup_{\alpha \in \mathcal{N}(k)} \left(\mathbf{F}_{k\alpha} \setminus (\mathbf{F}_{k\alpha} \ominus \mathbf{N}^{n-1}_k) \right), \tag{32}$$

$$(\mathbf{F} \oplus \mathbf{N}^n) \setminus \mathbf{F} = \bigcup_{k=1}^{n} \bigcup_{\alpha \in \mathcal{N}(k)} \left((\mathbf{F}_{k\alpha} \oplus \mathbf{N}^{n-1}_k) \setminus \mathbf{F}_{k\alpha} \right). \tag{33}$$

This lemma derives the following theorem.

Theorem 4. *The boundary* $\partial_\pm \mathbf{F}$ *of an n-dimensional digital object* $\mathbf{F}$ *is the union of its* $(n-1)$*-dimensional boundaries.*

Theorem 4 allows us to construct $\partial_\pm \mathbf{F}$ from $\partial_\pm \mathbf{F}_{k\alpha}$. Furthermore, Eqs. (10), (25) and (26) derive the relations

$$\mathbf{F}_{k\alpha} \setminus (\mathbf{F}_{k\alpha} \ominus \mathbf{N}^{n-1}_k) = \bigcup_{l=1}^{n-1} \bigcup_{\beta \in \mathcal{N}(l)} \left(\mathbf{F}_{k\alpha\, l\beta} \setminus (\mathbf{F}_{k\alpha\, l\beta} \ominus \mathbf{N}^{n-2}_{kl}) \right), \tag{34}$$

$$\left(\mathbf{F}_{k\alpha} \oplus \mathbf{N}^{n-1}_k \right) \setminus \mathbf{F}_{k\alpha} = \bigcup_{l=1}^{n-1} \bigcup_{\beta \in N(l)} \left(\left(\mathbf{F}_{k\alpha\, l\beta} \oplus \mathbf{N}^{n-2}_{kl} \right) \setminus \mathbf{F}_{k\alpha\, l\beta} \right). \tag{35}$$

Decomposing both a digital object and its neighbourhood by using Eqs. (10) and (26), respectively, we have the following theorem on the hierarchical construction method of boundary.

Theorem 5. *The boundary extraction methods are expressed in recursive forms*

$$\mathbf{F}_{k(l)\alpha(l)} \setminus (\mathbf{F}_{k(l)\alpha(l)} \ominus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l)}) = \bigcup_{k(l+1)=1}^{n-l} \bigcup_{\alpha(l+1)\in\mathcal{N}(k(l+1))} \left(\mathbf{F}_{k(l+1)\alpha(l+1)} \setminus (\mathbf{F}_{k(l+1)\alpha(l+1)} \ominus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l+1)})\right), \quad (36)$$

$$(\mathbf{F}_{k(l)\alpha(l)} \oplus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l)}) \setminus \mathbf{F}_{k(l)\alpha(l)} = \bigcup_{k(l+1)=1}^{n-l} \bigcup_{\alpha(l+1)\in\mathcal{N}(k(l+1))} \left((\mathbf{F}_{k(l+1)\alpha(l+1)} \oplus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l+1)}) \setminus \mathbf{F}_{k(l+1)\alpha(l+1)}\right), \quad (37)$$

for the internal and external digital boundaries, where $l = 0, 1, 2, \cdots, n-2$.

Using these relations in Theorem 5 recursively, we can construct the boundary-detection algorithm for n-dimensional digital objects from one-dimensional boundary detection algorithms.

Let $\mathbf{l}(k, \boldsymbol{\alpha}_k) = \{\boldsymbol{x} | \boldsymbol{x} = t\boldsymbol{e}_k + \sum_{i\neq k} \alpha_i \boldsymbol{e}_i\}$, for $\alpha_i \in \mathbf{Z}$, where $\alpha_k = \{\alpha_i\}_{i=1,\, i\neq k}^{n}$ and $\mathbf{F}_{k,\boldsymbol{\alpha}} = \mathbf{F} \bigcap \mathbf{l}(k, \boldsymbol{\alpha}_k)$. Then, by computing the one-dimensional internal and external boundaries $\partial_- \mathbf{F}_{k,\alpha} = \mathbf{F}_{k,\alpha_k} \setminus (\mathbf{F}_{k,\alpha_k} \ominus \mathbf{N}^1)$ and $\partial_+ \mathbf{F}_{k,\boldsymbol{\alpha}_k} = (\mathbf{F}_{k,\boldsymbol{\alpha}_k} \oplus \mathbf{N}^1) \setminus \mathbf{F}_{k,\boldsymbol{\alpha}_k}$, respectively, for all $\{\boldsymbol{e}_k, \boldsymbol{\alpha}_k\}_{k=1}^{n}$, we have the following theorem.

Theorem 6. *The n-dimensional internal and external boundaries are constructed as* $\partial_- \mathbf{F} = \cup_k \cup_{\alpha_k} \partial_- \mathbf{F}_{k,\alpha_k}$ *and* $\partial_+ \mathbf{F} = \bigcup_k \bigcup_{\boldsymbol{\alpha}_k} \partial_+ \mathbf{F}_{k,\boldsymbol{\alpha}_k}$, *respectively.*

There exist 3^n points in the region

$$\mathbf{C}(\boldsymbol{a}) = \{\boldsymbol{x} | |\boldsymbol{x} - \boldsymbol{a}|_\infty \leq 1\} \quad (38)$$

around a point $\boldsymbol{a}$. For the point $\boldsymbol{a} \in \mathbf{Z}^n$, if

$$(\partial_+ \mathbf{F} \cup \partial_- \mathbf{F}) \bigcap \mathbf{C}(a) = \mathbf{C}(a), \quad (39)$$

the point $\boldsymbol{a}$ lies on a locally flat manifold. Furthermore, if the point $\boldsymbol{a}$ lies on a corner, the relation

$$(\partial_+ \mathbf{F} \cup \partial_- \mathbf{F}) \bigcap \mathbf{C}(a) \subset \mathbf{C}(a) \quad (40)$$

is satisfied. Eqs. (38), (39) and (40) imply the following theorem on the geometric property of the corners.

Theorem 7. *On the corner the relation*

$$|(\partial_+ \mathbf{F} \bigcup \partial_- \mathbf{F}) \bigcap \mathbf{C}(a)| < 3^n \quad (41)$$

is satisfied.

The corner points of a digital object $\mathbf{F}$ may separate both the internal boundary $\partial_- \mathbf{F}$ and external boundary $\partial_- \mathbf{F}$ into portions. Using the corners of the internal and external boundaries of the complement $\overline{\mathbf{F}}$ of the digital object, we can refine the connectivity of both the internal and external boundaries.

Definition 22. If there exists at least one path between all pairs of points on the internal and external boundaries, these boundaries are called the refined internal and external boundaries, respectively.

Let $\mathbf{N}^2_{\alpha\beta}$ be the 2-dimensional neighbourhood on the digital plane parallel to the plane $\mathbf{Z}^2_{\alpha\beta} = \{\boldsymbol{x} | \boldsymbol{x} = \lambda \boldsymbol{e}_\alpha + \mu \boldsymbol{e}_\beta, \lambda, \mu \in \mathbf{Z}\}$. Refinement of $\mathbf{F}$ by $\mathbf{N}^2_{\alpha\beta}$ on the planes parallel to $\mathbf{Z}^2_{\alpha\beta}$ yields a set $\overline{\mathbf{F}_{\alpha\beta}}$. For appropriate combinations of pairs α and β, successive application of the refinement using the decomposed neighbourhood $\mathbf{N}^2_{\alpha\beta}$ transforms an imperfect set to a perfect set.

Definition 23. If a sequence of pairs $\{\langle \alpha_i, \beta_i \rangle\}_{i=1}^{k}$ are used to produce a refined set, we define the order of the operation to transform a point set to a perfect object

$$\overline{\mathbf{F}} := (\cdots((\mathbf{F}_{\alpha_1\beta_1})_{\alpha_2\beta_2})\cdots)_{\alpha_k\beta_k} \quad (42)$$

and the $\{\langle \alpha_i, \beta_i \rangle\}_{i=1}^{k}$ is the minimum sequence, we call the set $\mathbf{F}$ the k-refined set.

Theorem 8. *In* $\mathbf{Z}^n$, *the maximum length of the sequence for refinement is* $n-1$.

Proof. The length of minimum path from $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{e}$, where $\boldsymbol{e} = \sum_{i=1}^{n} \boldsymbol{e}_i$ is $n-1$, if we assume $2n$-neighbourhood. A series of the two-dimensional refinement operations using a sequence of the neighbourhoods $\{\mathbf{N}^2_{i\,i+1}\}_{i=1}^{n-1}$ decides a $2n$-connected path from the point $\boldsymbol{x}$ to the point $\boldsymbol{x} + \boldsymbol{e}$. □

Corollary 3. *In* $\mathbf{Z}^n$, *from*

$$\mathbf{F} = \left\{ \boldsymbol{x}, \boldsymbol{x} + \sum_{i=1}^{k} \boldsymbol{e}_i, k \geq 2 \right\}, \quad (43)$$

the k-refined set is yielded as

$$\mathbf{F} = \left\{ \boldsymbol{x},\ \boldsymbol{x} + \sum_{i=1}^{k} \boldsymbol{e}_i, \right\} \bigcup \left(\bigcup_{j=2}^{k} \left\{ \boldsymbol{x} + \sum_{i=1}^{j} \boldsymbol{e}_i \right\} \right), \quad (44)$$

using the sequebignce of neighbourhoods $\{\mathbf{N}^1_{\alpha\,\alpha+1}\}_{\alpha=1}^{n-1}$.

Example 1. In $\mathbf{Z}^4$ for

$$\mathbf{H} = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} a+1 \\ b+1 \\ c+1 \\ d+1 \end{pmatrix} \right\}, \quad (45)$$

using $\mathbf{N}^2_{12}$, $\mathbf{N}^2_{23}$ and $\mathbf{N}^2_{34}$, a 3-refined set

$$\overline{\mathbf{H}} = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} a+1 \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} a+1 \\ b+1 \\ c \\ d \end{pmatrix}, \begin{pmatrix} a+1 \\ b+1 \\ c+1 \\ d \end{pmatrix}, \begin{pmatrix} a+1 \\ b+1 \\ c+1 \\ d+1 \end{pmatrix} \right\} \quad (46)$$

is generated.

Next, we define the perfectness of objects in the k-sub-grid.

Definition 24. If the refinement of $\mathbf{F}^k$ in the k-sub-grid is perfect object, we call $\mathbf{F}$ is the k-perfect. For the perfectness of the $\mathbf{F}^k$ after refinement,

The definition of the perfectness implies that the relation $k \geq 3$ is required.

We call the point sets

$$\mathbf{C}_- = (\partial_- \overline{\mathbf{F}} \bigcup \partial_+ \mathbf{F}) \setminus (\partial_- \overline{\mathbf{F}} \bigcap \partial_+ \mathbf{F}), \quad (47)$$

$$\mathbf{C}_+ = (\partial_+ \overline{\mathbf{F}} \bigcup \partial_- \mathbf{F}) \setminus (\partial_+ \overline{\mathbf{F}} \bigcap \partial_- \mathbf{F}) \quad (48)$$

the singular points, which disturb the connectivity along the boundary curves. The refined internal and external boundaries

$$\overline{\partial_- \mathbf{F}} = \partial_- \mathbf{F} \cup \mathbf{C}_-, \quad \overline{\partial_+ \mathbf{F}} = \partial_+ \mathbf{F} \cup \mathbf{C}_+ \quad (49)$$

prevent the continuity. The relations in (49) are expressed as

$$\overline{\partial_- \mathbf{F}} = \partial_- \mathbf{F} \bigcup \partial_+ \overline{\mathbf{F}}, \qquad \overline{\partial_+ \mathbf{F}} = \partial_+ \mathbf{F} \bigcup \partial_- \overline{\mathbf{F}}. \quad (50)$$

Fig. 4 illustrates the refinement operation for boundary detection. Refinement operations at the corners preserve the continuity of the internal and external boundary.

$\partial_\pm \overline{\mathbf{F}}$ is numerically computed by $\partial_\pm \overline{\mathbf{F}} = \{\partial_\pm (\mathbf{H} \setminus \mathbf{F})\} \setminus \partial_\pm \mathbf{F}$ for a large hypercube $\mathbf{H}$, which encloses $\mathbf{F}$ with the condition

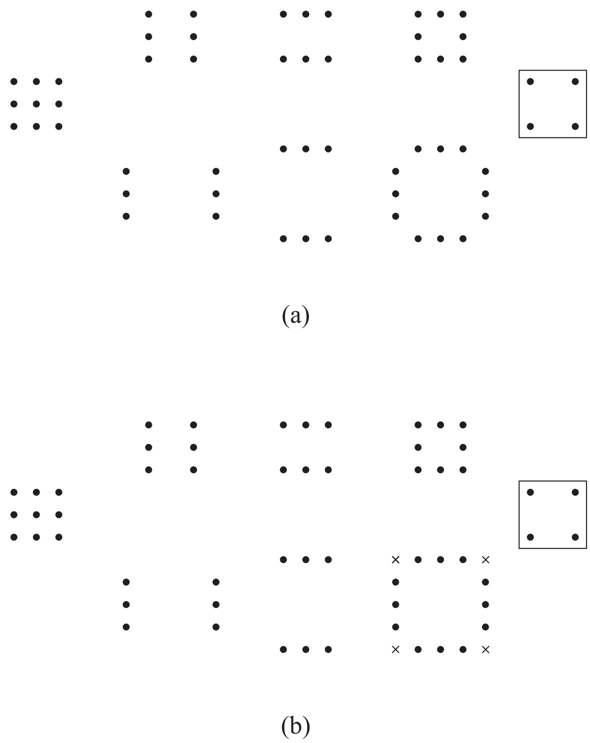

Fig. 4. Refinement operation and boundary detection. (a) Union of the internal and external boundaries. (b) Refinement operations at the corners preserve the continuity of the internal and external boundaries.

$\min_{\boldsymbol{x}\in(\mathbf{H}\setminus\mathbf{F}),\boldsymbol{y}\in\mathbf{F}}|\boldsymbol{x}-\boldsymbol{y}|\geq 3$ on the isothetic lines $\boldsymbol{z}=\boldsymbol{a}+t\boldsymbol{e}_i$ for $\boldsymbol{a}\in\mathbf{Z}^n$.

Definition 25. The digital set gradient on the boundary is

$$\partial\mathbf{F}=\left(\bigcup_{\boldsymbol{x}\in\overline{\partial_+\mathbf{F}}}\mathbf{V}(\boldsymbol{x})\right)\bigcap\left(\bigcup_{\boldsymbol{x}\in\overline{\partial_-\mathbf{F}}}\mathbf{V}(\boldsymbol{x})\right). \tag{51}$$

$\partial\mathbf{F}$ is the boundary of the embedding $\mathcal{F}$ of the object $\mathbf{F}$, that is, $\partial\mathbf{F}=\partial\mathcal{F}$. Then, $\partial\mathbf{F}$ is an isothetic Nef-polytope [11] whose vertices and faces lie on the dual grid $\mathbf{D}^n$. Therefore, we have the next lemma.

Lemma 5. *$\partial\mathcal{F}$ is a union of $(n-1)$ simplices [16] in the dual grid.*

Proof. All vertices of $\partial\mathcal{F}$ lie on the dual grid $\mathbf{D}^n$. Furthermore, the collection of vertices of the boundary of each voxel is $(n-1)$-simplex on $\mathbf{D}^n$. □

Let $[\partial\mathbf{F}]=[\partial\mathcal{F}]$ be the closure of $\partial\mathbf{F}=\partial\mathcal{F}$. $[\partial\mathbf{F}]=[\partial\mathcal{F}]$ satisfy the next lemma.

Lemma 6. *The closure of $[\partial\mathbf{F}]=[\partial\mathcal{F}]$ is an n-complex in the dual grid.*

Proof. All vertices of the closure of $[\partial\mathbf{F}]$ lie on the dual grid $\mathbf{D}^n$. Since the collection of vertices of each voxel is a simplex in $\mathbf{D}^n$, $[\partial\mathbf{F}]=[\partial\mathcal{F}]$ is a connected union of digital n-simplices defined in $\mathbf{D}^n$. □

For the thickness of $[\partial\mathbf{F}]=[\partial\mathcal{F}]$, we have the next theorem.

Theorem 9. *The thickness of the complement of $[\partial\mathbf{F}]=[\partial\mathcal{F}]$ is at least two voxels.*

Proof. On any isothetic digital line $\mathbf{L}(k,\boldsymbol{z})=\lambda\boldsymbol{e}_k+\boldsymbol{z}$ for $\boldsymbol{z}\in\mathbf{Z}^n$ parallel to the vector $\boldsymbol{e}_k$, the linear object $\mathbf{F}(k,\boldsymbol{z})=\mathbf{F}\bigcap\mathbf{L}(k,\boldsymbol{z})$ is a thick one-dimensional object. The thickness of the complement of the embedding $\mathcal{F}(k,\boldsymbol{z})$ is at least two voxels. □

Theorem 9 implies the following statement on the embedding of digital objects in a digital space into Euclidean space.

Theorem 10. *An isothetic Nef-polytope $\mathcal{F}$ and its complement are perfect voxel objects.*

6. Distance transform, skeleton set and thinning

For integer d, the distance set $\mathbf{F}(d)$ of $\mathbf{F}\in\mathbf{Z}^n$ is defined as following.

Definition 26.

$$\mathbf{F}(d)=\begin{cases}\{\boldsymbol{x}|\min_{\boldsymbol{x}\in\mathbf{F},\boldsymbol{y}\in\overline{\mathbf{F}}}|\boldsymbol{x}-\boldsymbol{y}|_\infty=d\}, & d>0,\\ \{\boldsymbol{x}|\min_{\boldsymbol{y}\in\mathbf{F},\boldsymbol{x}\in\overline{\mathbf{F}}}|\boldsymbol{x}-\boldsymbol{y}|_\infty=d\}, & d<0.\end{cases} \tag{52}$$

The negative value of the distance in Eq. (52) expresses the distance between an object, which is a connected component, and a point which are not connected to the object. Since Definitions 21 and 26 imply the relations $\mathbf{F}(1)=\partial_-\mathbf{F}$ and $\mathbf{F}(-1)=\partial_+\mathbf{F}$. the recursive forms

$$\mathbf{F}(d+1)=\mathbf{F}(d)\setminus(\mathbf{F}(d)\ominus\mathbf{N}^n),\ \mathbf{F}(0):=\mathbf{F},\ d\geq 0, \tag{53}$$

$$\mathbf{F}(d-1)=(\mathbf{F}(d)\oplus\mathbf{N}^n)\setminus\mathbf{F}(d),\ \mathbf{F}(0):=\mathbf{F},\ d\leq 0. \tag{54}$$

compute the distance set $\mathbf{F}(d)$ of $\mathbf{F}\in\mathbf{Z}^n$. The following algorithm achieves the distance transform for $d>0$.

For the positive label d, labelling d to each point in $\mathbf{F}(d)$, the distance map $D_{\mathbf{F}}(\boldsymbol{x})$, $D_{\mathbf{F}}(\boldsymbol{x})|_{\boldsymbol{x}\in\mathbf{F}(d)}=d$ of $\mathbf{F}$ is constructed. $d_{\max}$ is detected, when algorithm 1 halts.

Algorithm 1: Distance transform for $d>0$.

Data: **F**
Result: $\{\mathbf{F}(d)\}_{d=1}^{d_{\max}}$
$d:=0$, $\mathbf{F}(0):=\mathbf{F}$;
while $\mathbf{F}(d+1)\neq\emptyset$ **do**
  $\mathbf{F}(d+1)=\mathbf{F}(d)\setminus(\mathbf{F}(d)\ominus\mathbf{N}^n)$;
  $d:=d+1$;

The distance map $D_{\mathbf{F}}(\boldsymbol{x})$ derives the distance-based skeleton set as

$$\mathbf{S}_{\mathbf{F}}=\bigcup_{d=1}^{d_{\max}}\{\boldsymbol{x}|d=\max_{\boldsymbol{y}\in\mathbf{N}^n(\boldsymbol{x})}D_{\mathbf{F}}(\boldsymbol{y})\}, \tag{55}$$

that is, the distance-based skeleton set is the collection of the local maximal points of the distance map. Since perfectness is not guaranteed for points in $\mathbf{S}_{\mathbf{F}}$, application of dilation to $\mathbf{S}_{\mathbf{F}}$ derives a connected set

$$\mathbf{T}_{\mathbf{F}}=\mathbf{S}_{\mathbf{F}}\oplus\mathbf{N}^n=\mathbf{S}_{\mathbf{F}}\bigcup\mathbf{S}_{\mathbf{F}}(-1). \tag{56}$$

We call $\mathbf{T}_{\mathbf{F}}$ the trunk of $\mathbf{F}$. For the trunk we have the following.

Theorem 11. *The trunk $\mathbf{T}_{\mathbf{F}}$ of $\mathbf{F}\in\mathbf{Z}^n$ is a thin object.*

Proof. On the intersections of $\mathbf{S}_{\mathbf{F}}$ and a pair of adjacent parallel lines $\mathbf{l}_0=t\boldsymbol{e}_k+\alpha\boldsymbol{e}_l$ and $\mathbf{l}_+=t\boldsymbol{e}_k(\alpha+1)\boldsymbol{e}_l$ for $t\in\mathbf{Z}$, where $k\neq l$, for a pair of successive integers α and $\alpha+1$, three local configurations

$$\begin{matrix}\circ & \bullet\\ \circ & \bullet\end{matrix}\quad\begin{matrix}\circ & \circ\\ \circ, & \circ\end{matrix}\quad\begin{matrix}\circ & \bullet\\ \bullet & \circ\end{matrix}\quad\begin{matrix}\circ & \bullet\\ \circ, & \circ\end{matrix}\quad\begin{matrix}\circ & \circ\\ \bullet & \circ,\end{matrix}$$

exit, where the points with labels $\bullet$ and $\circ$ belong to $\mathbf{S}_{\mathbf{F}}$ and its compliment.

One dimensional dilation along lines $\mathbf{l}_\circ$ and $\mathbf{l}_+$ derives the local configurations

$$\begin{matrix}\bullet & \bullet\\ \bullet & \bullet\end{matrix}\quad\begin{matrix}\bullet & \circ\\ \bullet, & \bullet\end{matrix}\quad\begin{matrix}\bullet & \bullet\\ \bullet & \bullet\end{matrix}\quad\begin{matrix}\bullet & \bullet\\ *, & \circ\end{matrix}\quad\begin{matrix}\bullet & \bullet & *\\ \bullet & \bullet & \bullet,\end{matrix}$$

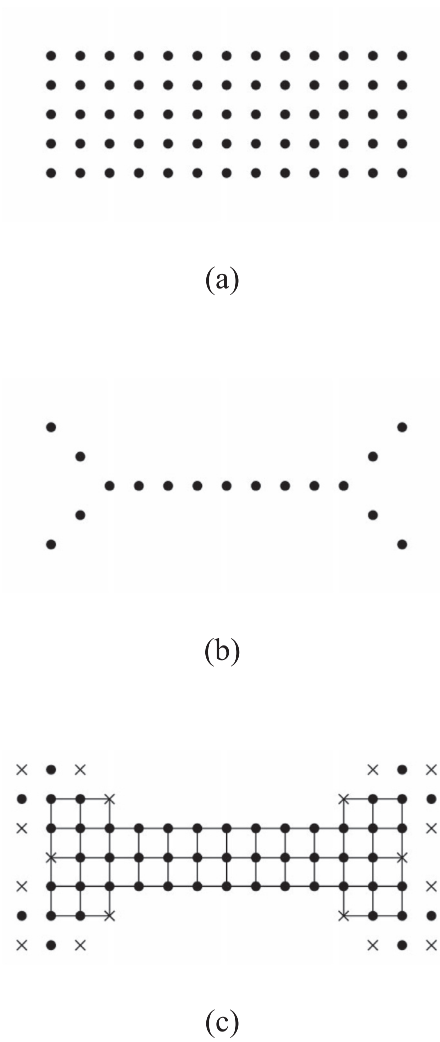

Fig. 5. Skelton set and the r-trunk on a point set. From top to down, (a) a point set, (b) the skeleton set of (a) and (c) r-trunk of the set (a). The refinement yields points with the mark $\times$. The points $\bullet$ without connecting bars are points which are not included to the original point set.

where $*$ is $\bullet$ or $\circ$ depending on local structures of $\mathbf{S}_\mathbf{F}$. These configuration imply that $\mathbf{S}_\mathbf{F} \oplus \mathbf{N}^1_k$ is locally thin. Since $\mathbf{S}_\mathbf{F} \oplus \mathbf{N}^1_k$ is locally an concatenations of these configurations in the direction of $\{\boldsymbol{e}_i\}_{i \neq k}$, $\mathbf{S}_\mathbf{F} \oplus \mathbf{N}^1_k$ is thin for $k = 1, 2, \cdots, n$. Therefore,

$$\mathbf{S}_\mathbf{F} \oplus \mathbf{N}^n = \bigcup_{k=1}^{n} \mathbf{S}_\mathbf{F} \oplus \mathbf{N}^1_k \tag{57}$$

is thin. □

The refinement operation to trunk $\mathbf{T}_\mathbf{F}$ of $\mathbf{F}$ yields a thick object.

Definition 27. The r-trunk of a perfect digital object $\mathbf{F}$ is

$$r\mathbf{T}_\mathbf{F} = \overline{\mathbf{S}_\mathbf{F} \oplus \mathbf{N}^n} \bigcap \mathbf{F}. \tag{58}$$

The r-trunk which expresses the skeletal object of a point set in $\mathbf{Z}^n$ possess the next theorem.

Theorem 12. *The r-trunk of a perfect digital object $\mathbf{F}$ is perfect.*

Fig. 5(a)–(c) show a point set, its skeletal set and the r-trunk extracted from the point set (a). The refinement yields points with the mark $\times$. The points $\bullet$ without connecting bars are points which are not included to the original point set.

Eqs. (53) and (54) imply the following theorem.

Theorem 13. *Decomposition of objects and neighbourhood derives the relations*

$$\mathbf{F}_{k(l)\alpha(l)}(d+1) = \bigcup_{k(l+1)=1}^{n-l} \bigcup_{\alpha(l+1)\in\mathcal{N}(k(l+1))} \Big(\mathbf{F}_{k(l+1)\alpha(l+1)}(d) \setminus (\mathbf{F}_{k(l+1)\alpha(l+1)}(d) \ominus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l+1)})\Big), \tag{59}$$

$$\mathbf{F}_{k(l)\alpha(l)}(d-1) = \bigcup_{k(l+1)=1}^{n-l} \bigcup_{\alpha(l+1)\in N(k(l+1))} \left(\left(\mathbf{F}_{k(l+1)\alpha(l+1)}(d) \oplus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l+1)}\right) \setminus \mathbf{F}_{k(l+1)\alpha(l+1)}(d)\right), \tag{60}$$

for computation of the distance sets and truck objects.

Using decomposition of the Minkowski subtraction, the hit-or-miss transform is decomposed.

Corollary 4. *With the condition $\mathbf{G} \cap \mathbf{H} = \emptyset$, which implies $\mathbf{G}_{k\alpha} \cap \mathbf{H}_{k\alpha} = \emptyset$, the hit-or-miss transform is expressed as*

$$\begin{aligned}(\mathbf{F} \ominus \mathbf{G}) \bigcap (\bar{\mathbf{F}} \ominus \mathbf{H}) &= (\bigcup_{k=1}^{n} \bigcup_{\alpha\in\mathcal{N}(k)} \mathbf{F}_{k\alpha} \ominus \mathbf{G}_{k\alpha}) \bigcap (\bigcup_{l=1}^{n} \bigcup_{\beta\in\mathcal{N}(k)} \mathbf{F}_{k\beta} \ominus \mathbf{G}_{k\beta}) \\ &= \bigcup_{k=1}^{n} \bigcup_{\alpha\in\mathcal{N}(k)} [(\mathbf{F}_{k\alpha} \ominus \mathbf{G}_{k\alpha}) \bigcap (\overline{\mathbf{F}_{k\alpha}} \ominus \mathbf{H}_{k\alpha})].\end{aligned} \tag{61}$$

This expression implies that operations for the hit-or-miss transform to the point set $\mathbf{F}$ and its compliment $\bar{\mathbf{F}}$ are decomposed into those in the lower-dimensional spaces.

The iteration form

$$\mathbf{F}^{(i+1)} = \mathbf{F}^{(i)} \setminus [(\mathbf{F}^{(i)} \ominus \mathbf{G}) \bigcap (\overline{\mathbf{F}^{(i)}} \ominus \mathbf{H})], \tag{62}$$

where $\mathbf{F}^{(0)} := \mathbf{F}$, for $i \geq 0$, achieves thinning of point set $\mathbf{F}$ by selecting an appropriate series of pairs $\mathbf{G}$ and $\mathbf{H}$. This expression of thinning and Corollary 4 for the hit-or-miss transform derive multislice and multidirectional decomposition of thinning.

Theorem 14. *The operation of Eq. (62) for thinning is decomposed as*

$$\mathbf{F}^{(i+1)} = \bigcup_{k=1}^{n} \bigcup_{\alpha\in\mathcal{N}(k)} \{\mathbf{F}^{(i)}_{k\alpha} \setminus [(\mathbf{F}^{(i)}_{k\alpha} \ominus \mathbf{G}_{k\alpha}) \bigcap (\overline{\mathbf{F}^{(i)}_{k\alpha}} \ominus \mathbf{H}_{k\alpha})]\}. \tag{63}$$

Theorem 14 leads to the conclusion that thinning is achieved slice by slice in each direction by selecting pairs of elements for the hit-or-miss transform.

7. Conclusions

We showed an explicit decomposition geometry of the neighbourhood in higher-dimensional digital space. This decomposition property clarifies that morphological operations in an n-dimensional digital space can be computed as the union of lower-dimensional morphological operations on isothetic digital subspaces intersecting with the digital objects. Decomposition procrdures of motphological operations provide us to construct new operations by the combination of the well-established operations in low-dimensional spaces.

Construction properties of the neighbourhood allow us to estimate the upper bound of operation times from the combination of operations in lower-dimensional spaces. Let T_k be the computational time of an operation in k dimensional space for $k \leq n$. The decomposition of operations to the lower-dimensional ones implies the relation $T_n \leq s_k \times {}_nC_k \times T_k$ where s_k and ${}_nC_k$ are the maximum number of the $(n-k)$-dimensional slices and the number of decompositions of object for operations to lower-dimensional spaces. If $k = n$, then $s_k = 1$ and ${}_nC_k$.

In a perfect voxel object, any voxels are contained as connected components, since any perfect voxel objects are Euclidean embedding of thick objects in $\mathbf{Z}^n$. The well-composdness [17–19] preserve connectives of Euclidian embedding of thin digital objects. A reformulation of well-composedness from the view point of thickness and thinness of voxel objects using resampling in k-subgrid is a future work.

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