Numerical resolution of Monge-Ampère equations arising in optics

Boris Thibert

Joint works with Quentin Mérigot and Jocelyn Meyron

Digital Geometry and Discrete Variational Calculus
Luminy (CIRM) - April 1 2021

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April fool's joke !

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Nonimaging optics: motivations

Goal: design components that transfer a prescribed light source
to a prescribed target illumination

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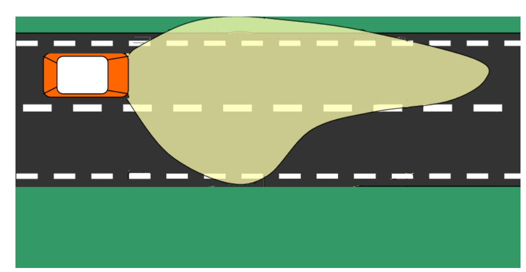
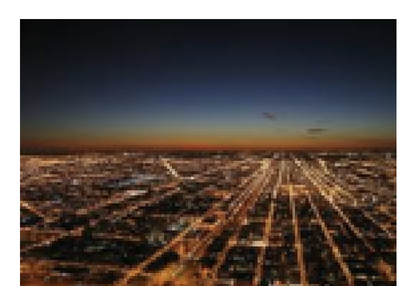
Motivations / applications

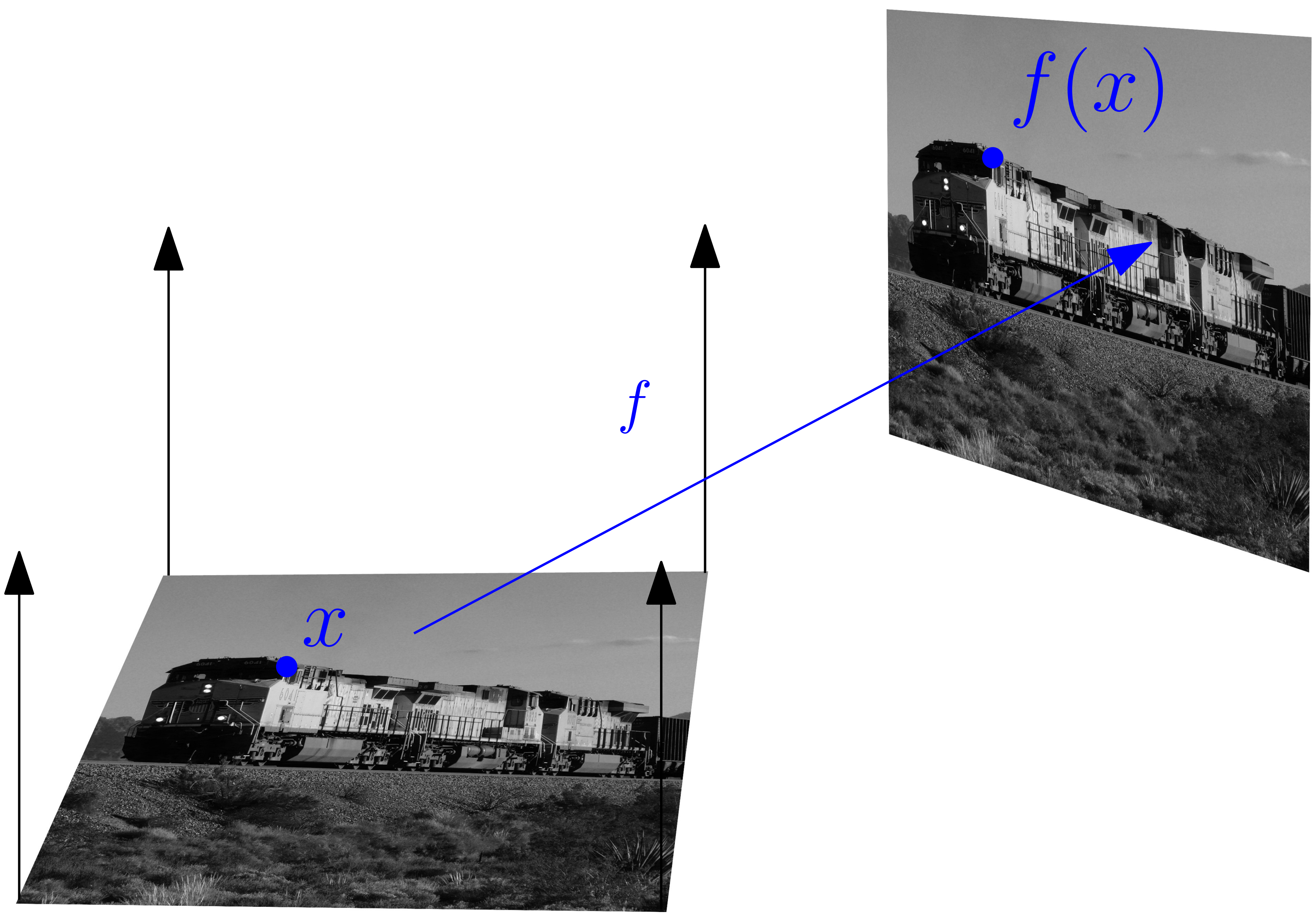
- Car beam design
- Public lighting: stadium, streets,...
- Reduction of light pollution





Imaging optics: mirror case

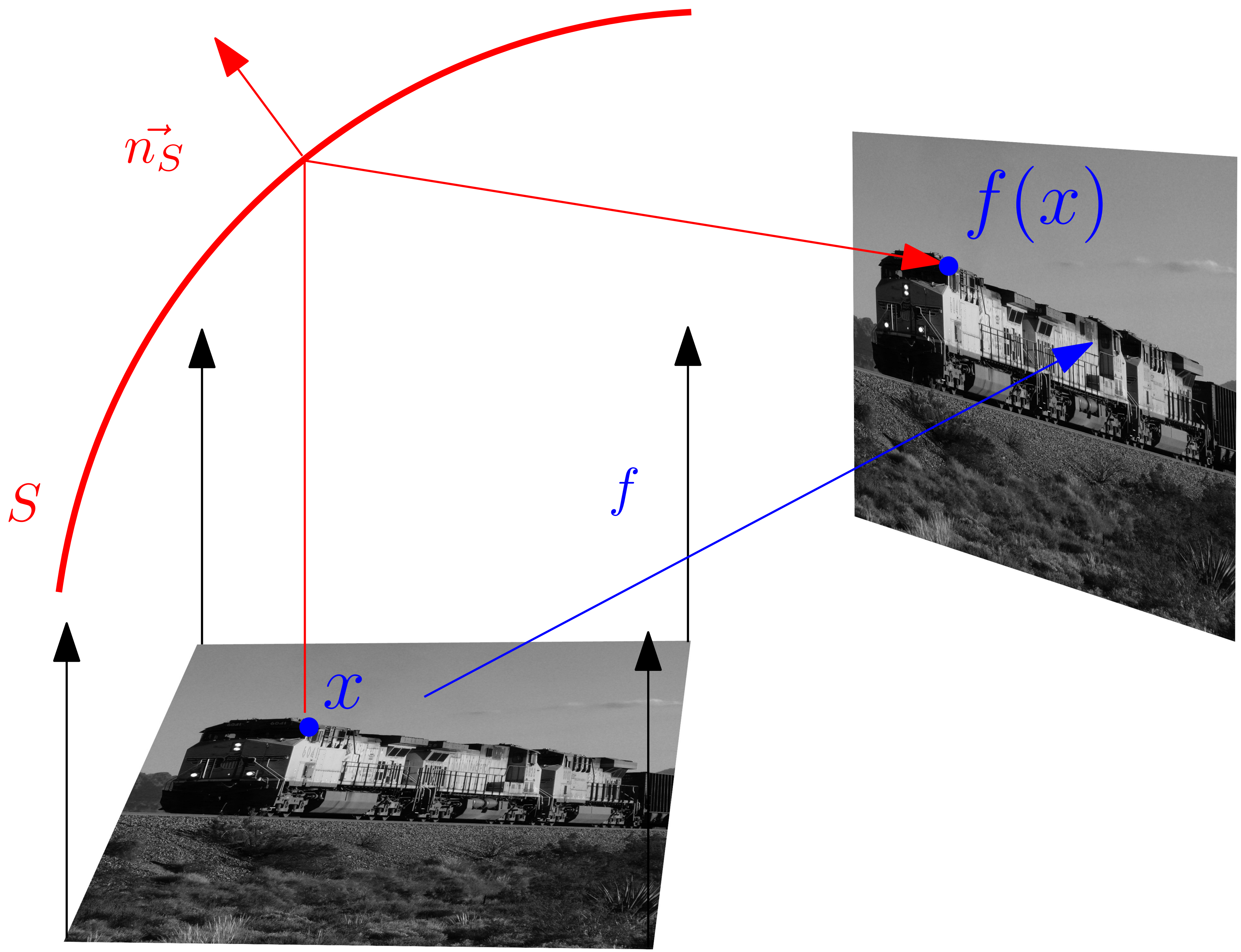
We are given a one-to-one map $f : X \to Y$.



Imaging optics: mirror case

We are given a one-to-one map $f : X \to Y$.

Goal: Find a surface S such that the reflection of X onto Y preserves f.

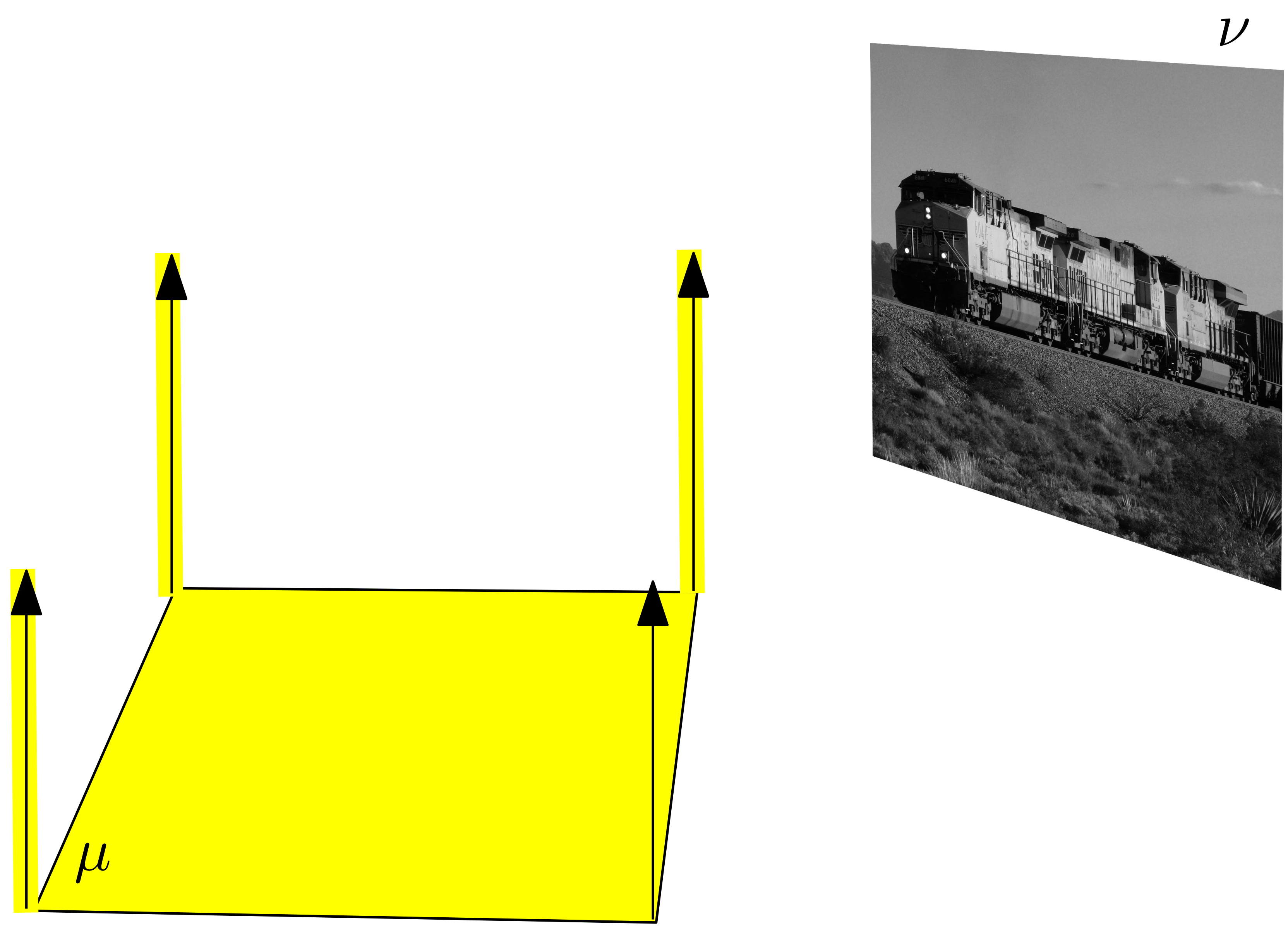


Non-imaging optics: mirror case

Input: Light source with intensity μ

Target illumination with intensity ν

No one-to-one map given



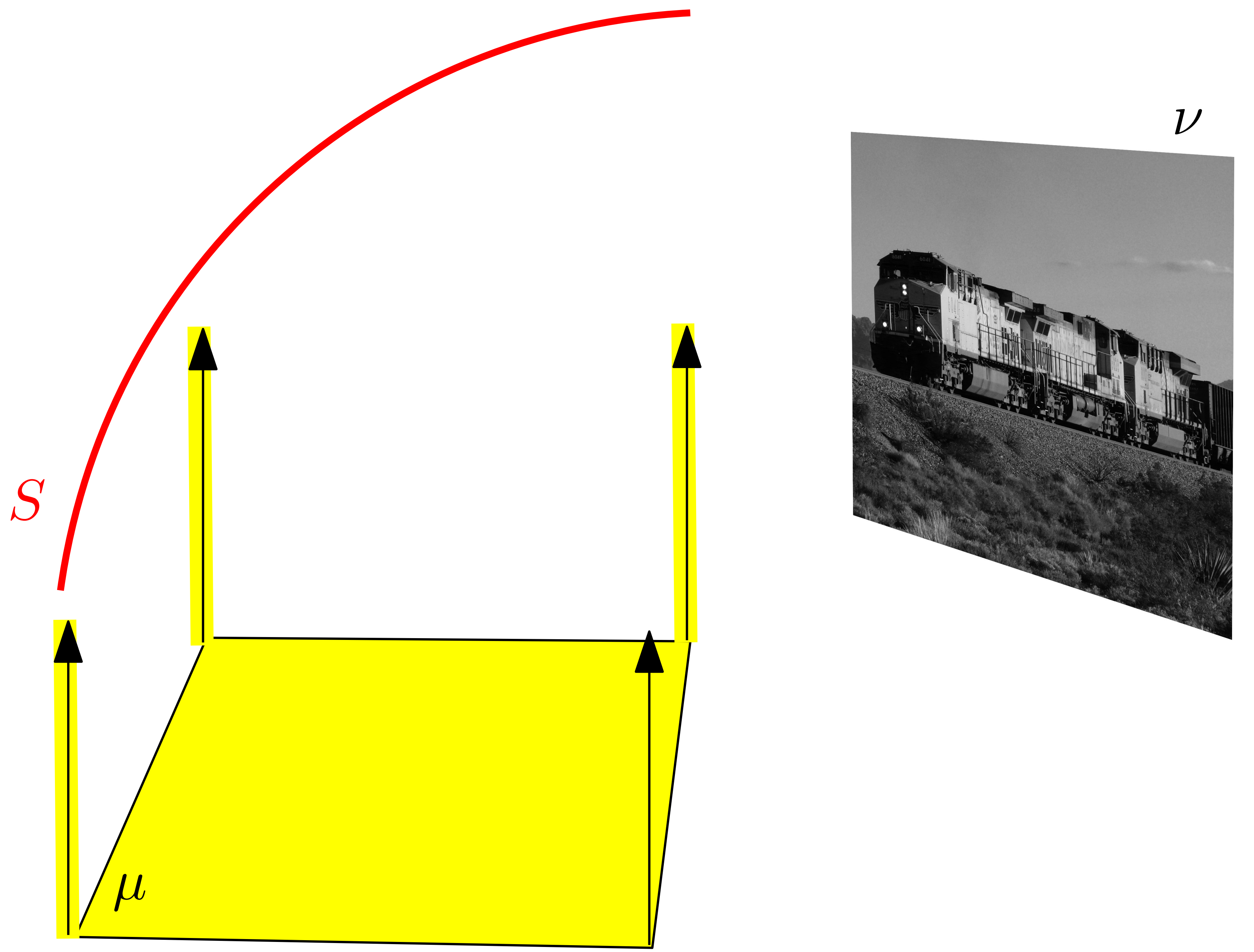
Non-imaging optics: mirror case

Input: Light source with intensity μ

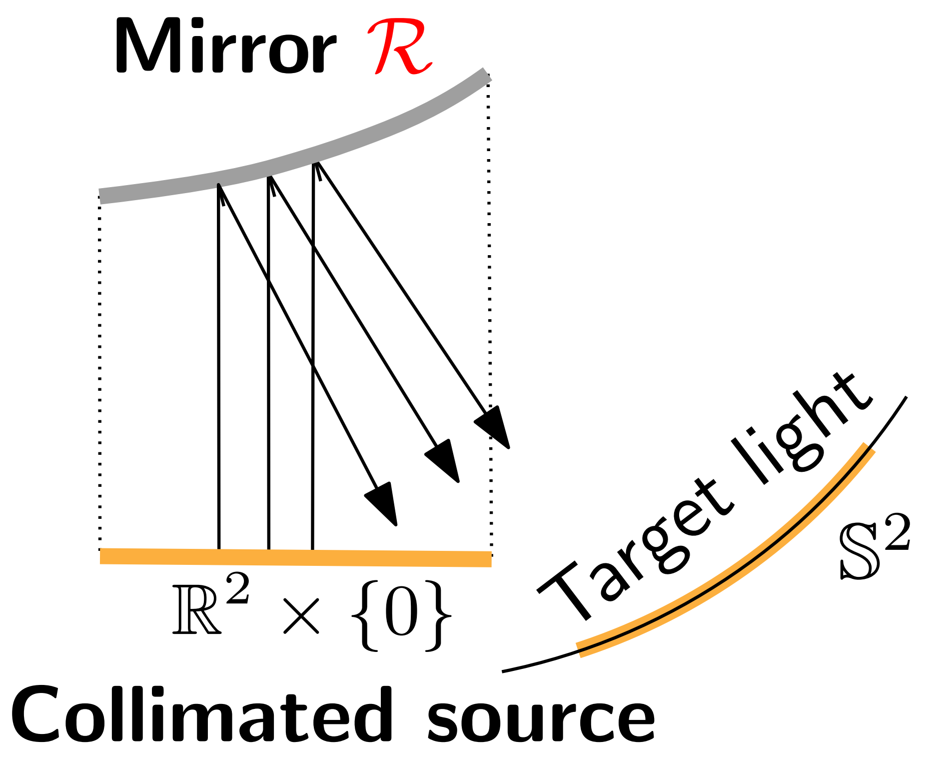
Target illumination with intensity ν

No one-to-one map given

Goal: Find a surface S such that reflects μ to the ν by Snell's law



Four inverse problems



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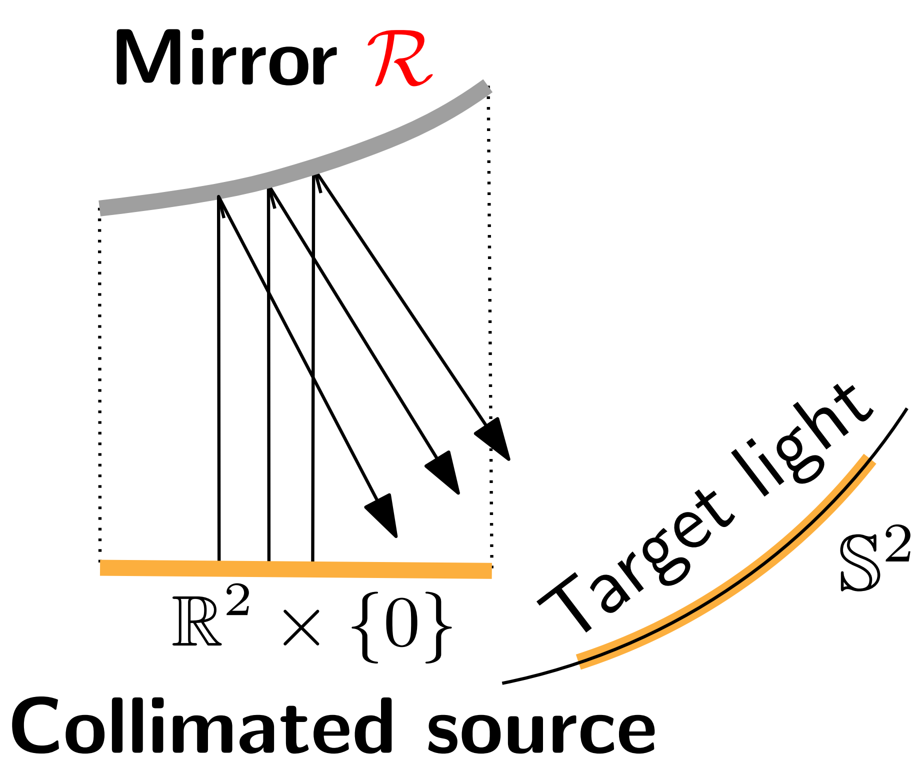

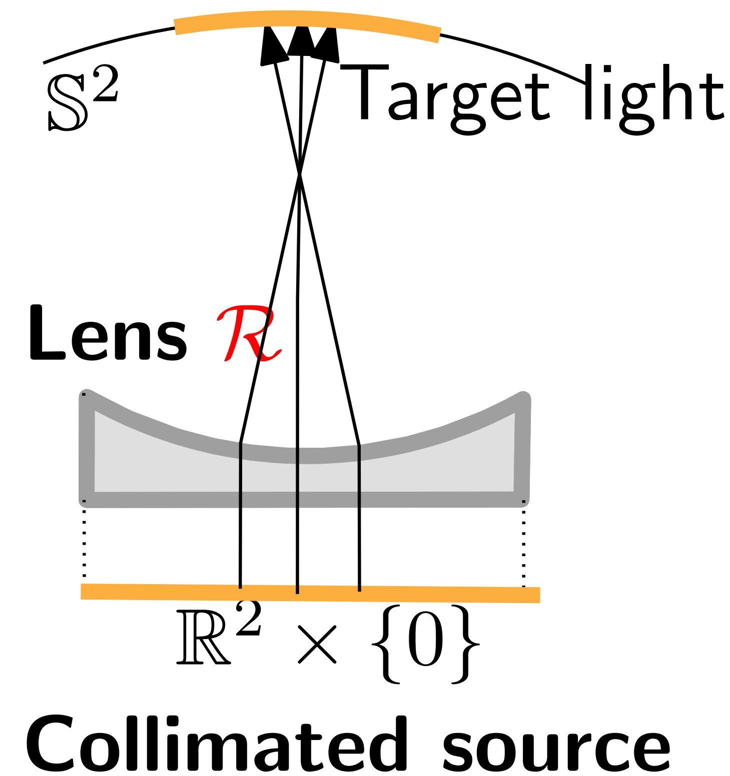

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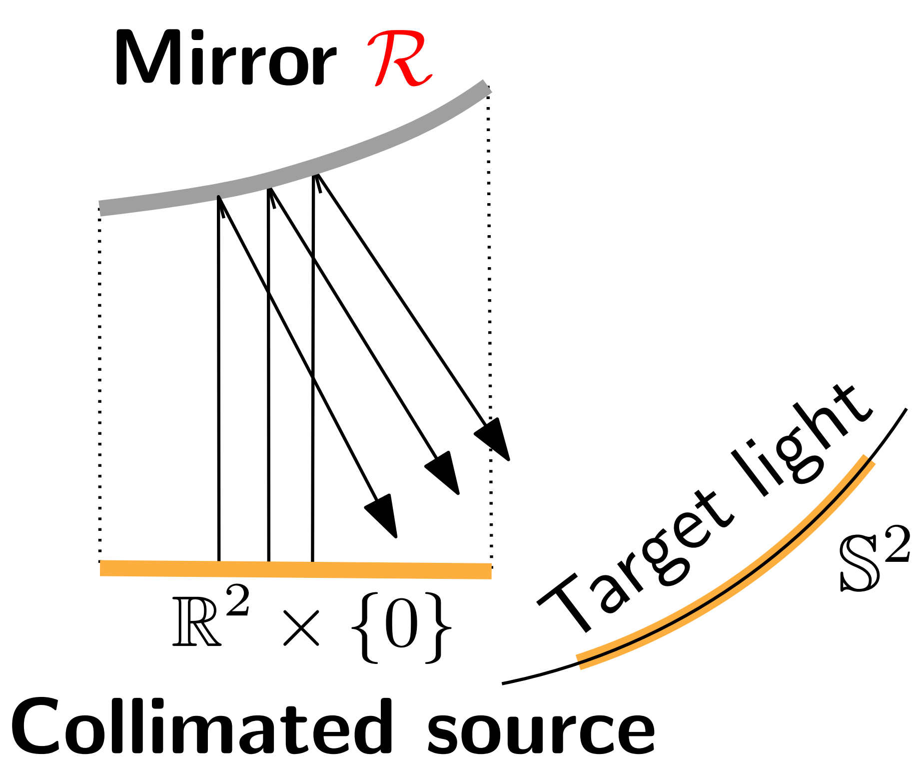

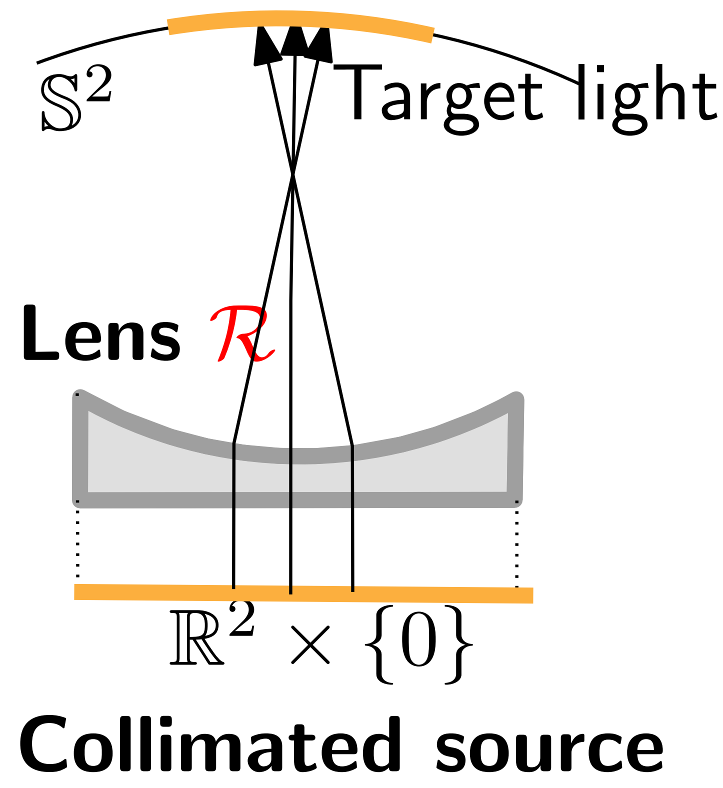

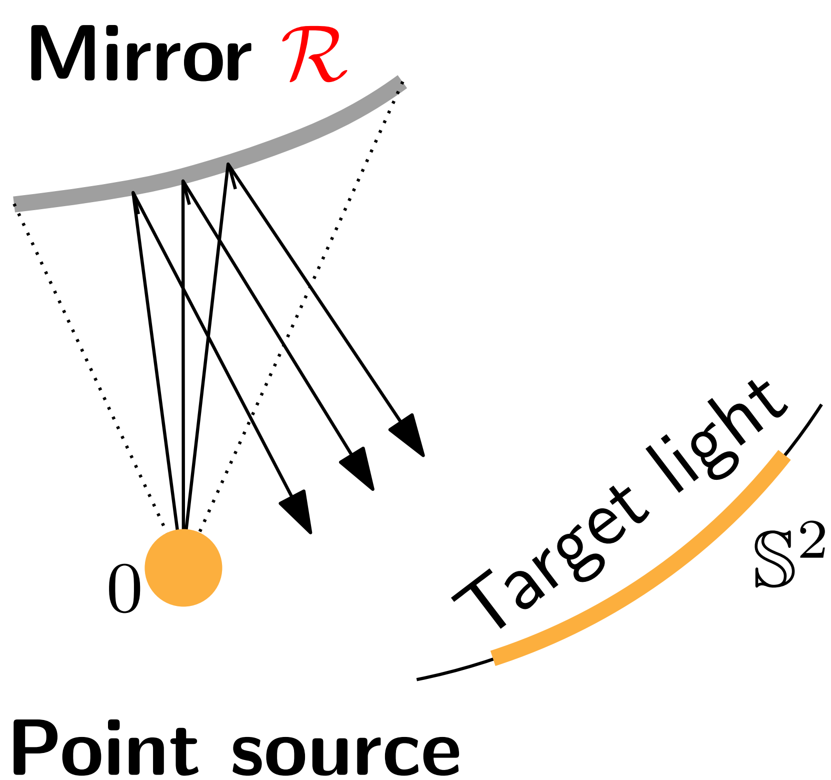

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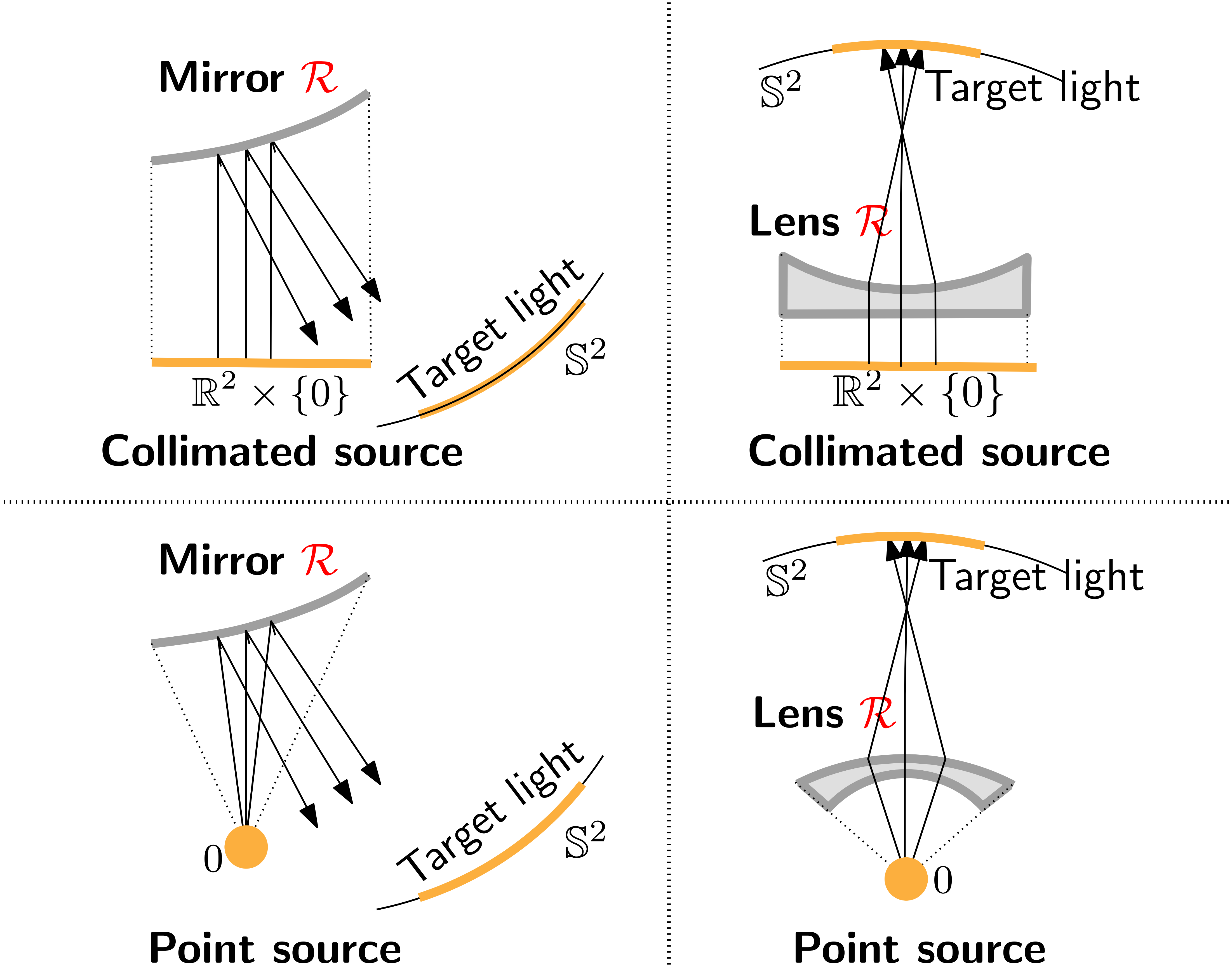

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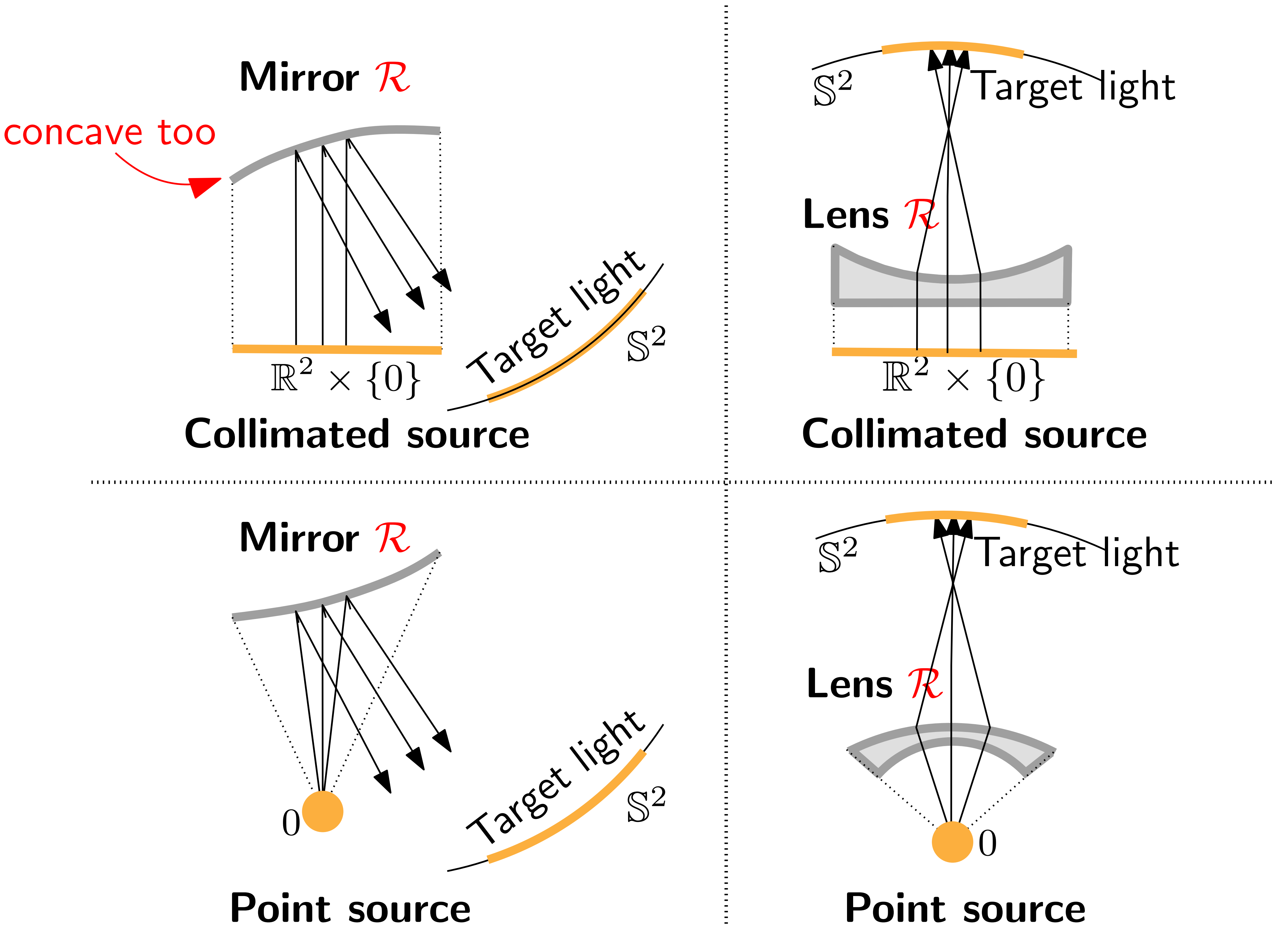


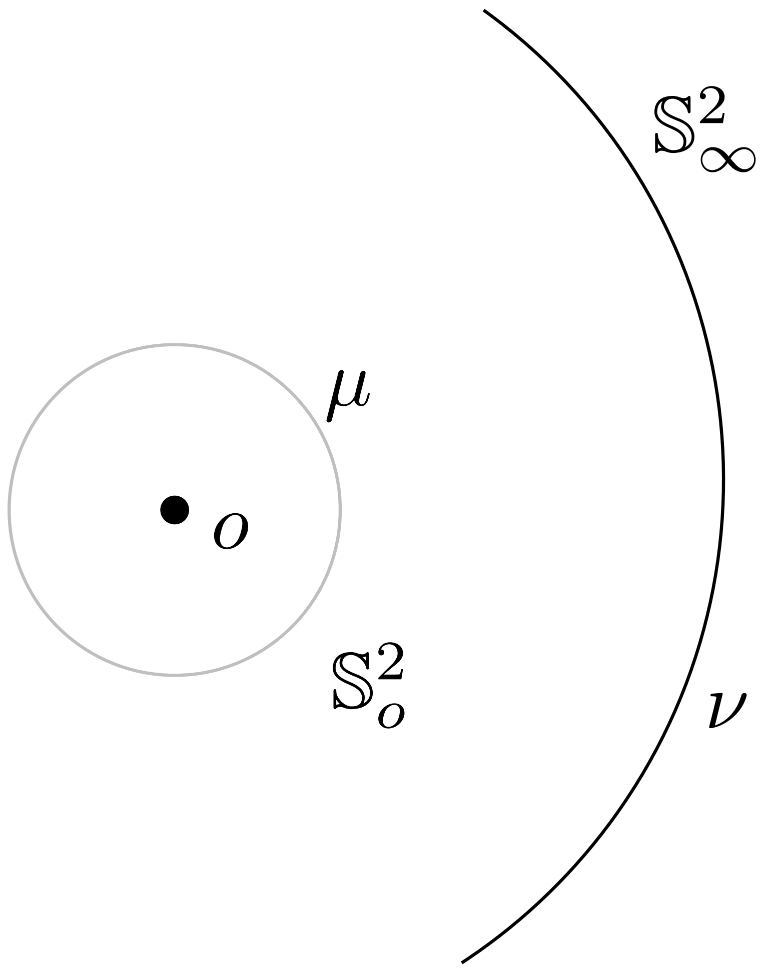
Outline

- Case 1: mirror for point light source
- Case 2: mirror for collimated light source

- Optimal transport
- Semi-discrete optimal transport
- Damped Newton algorithm

- Non-imaging optics: Far-Field target
- Non-imaging optics: Near-Field target

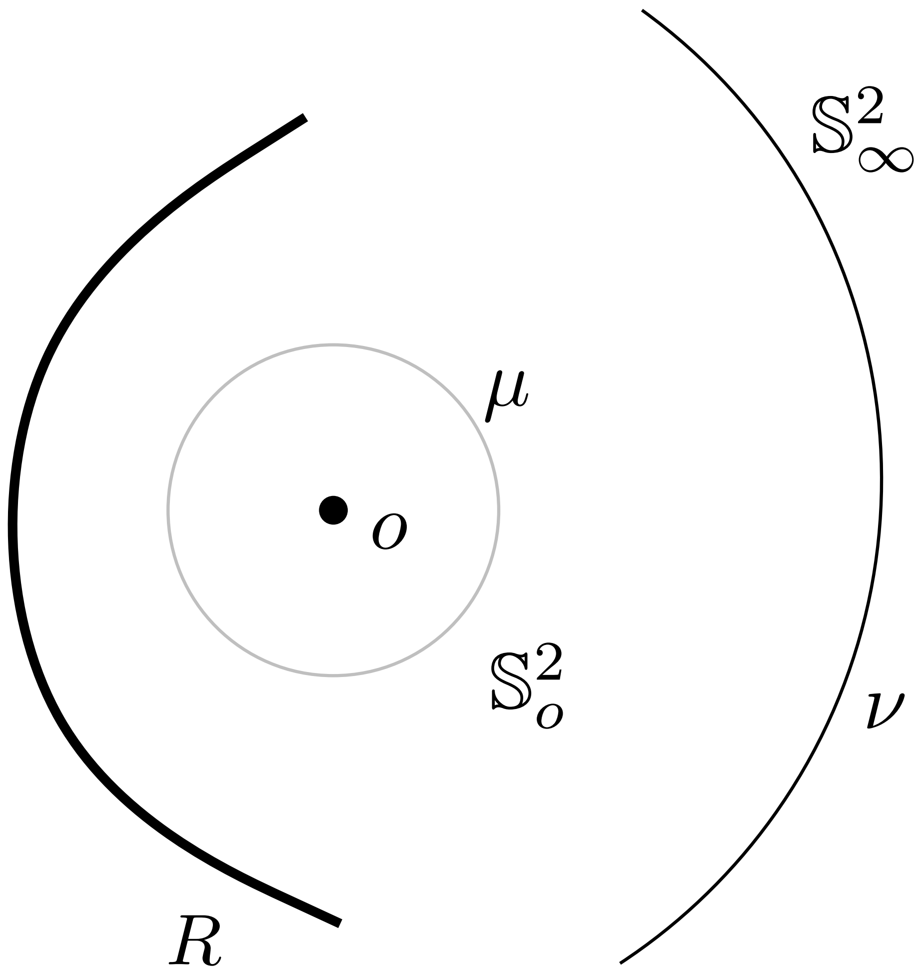
Mirror / Point light source



Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

Prescribed far-field: ν on $\mathbb{S}^2_\infty$

Mirror / Point light source



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Goal: Find a surface R which sends $(\mathbb{S}^2_o, \mu)$ to $(\mathbb{S}_\infty, \nu)$ under reflection by Snell's law.

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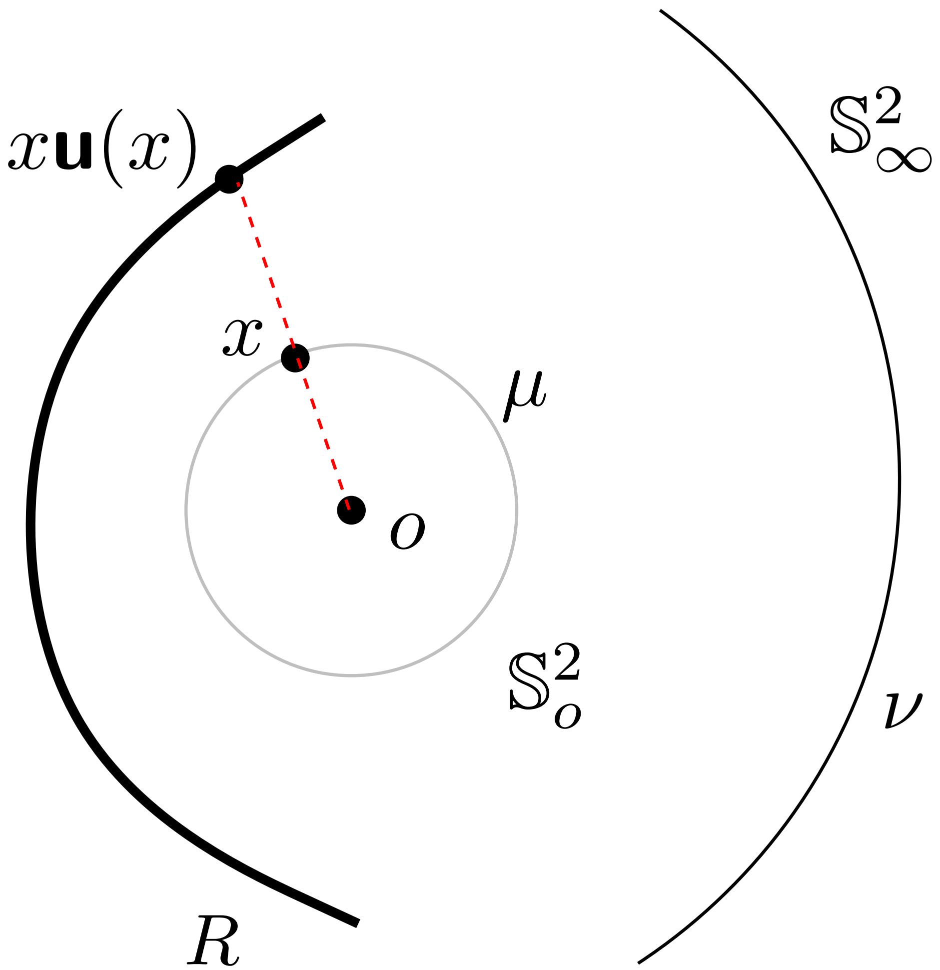

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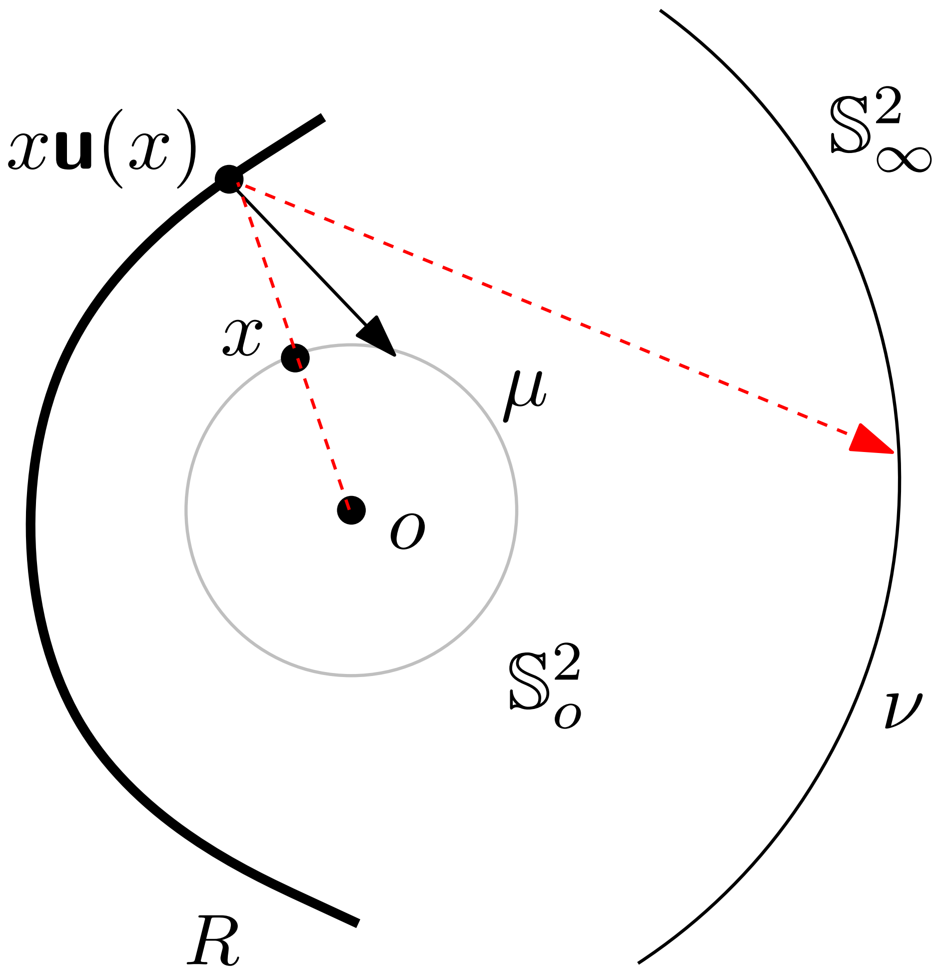

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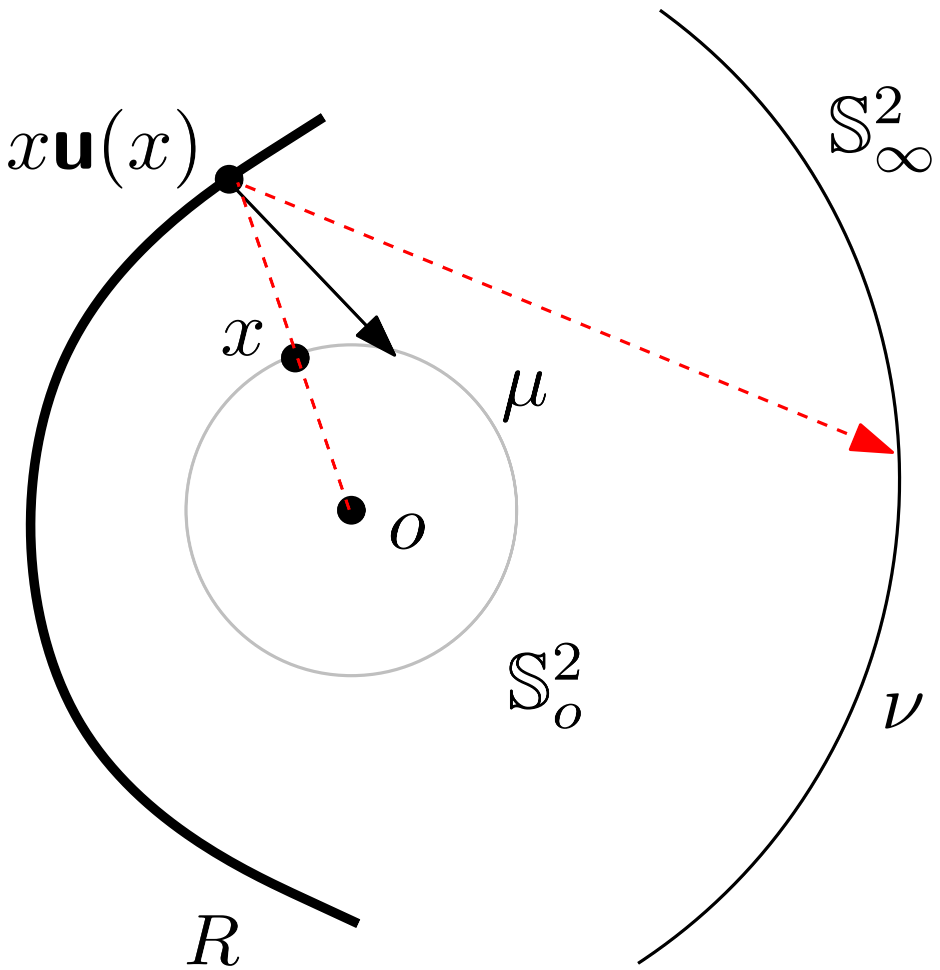

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Brenier formulation $T_\sharp \mu = \nu$

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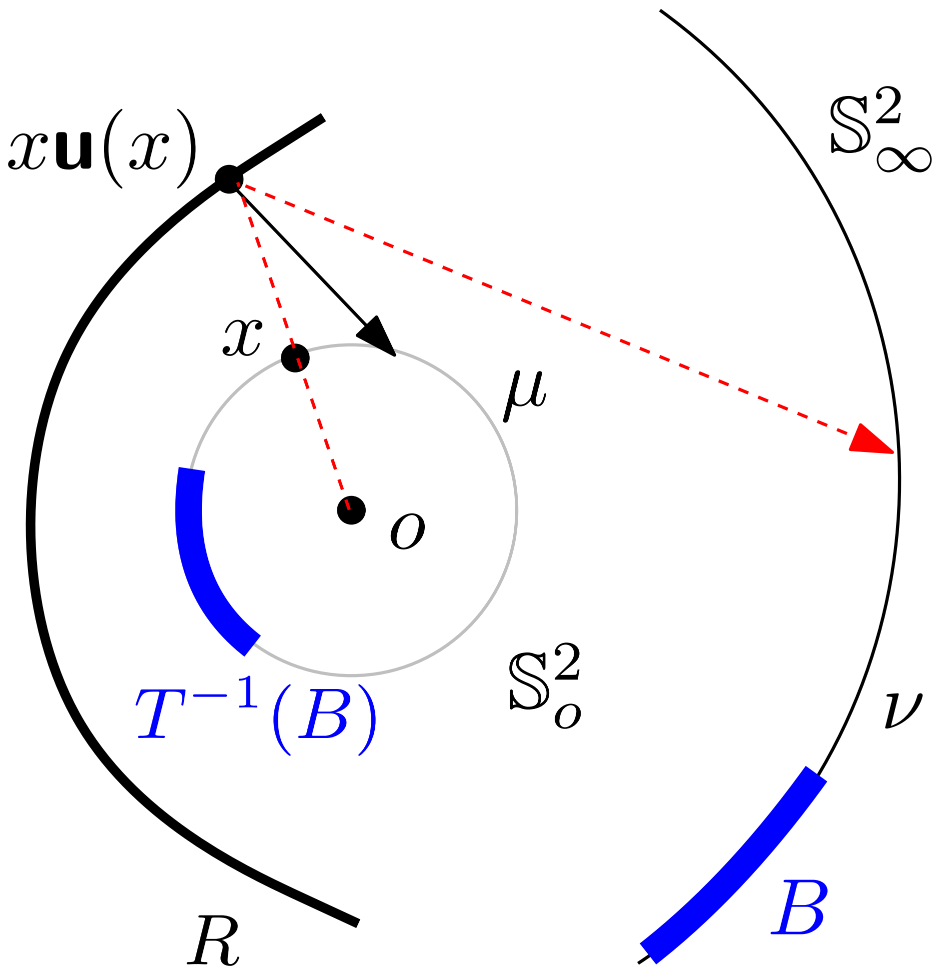

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i.e. for every borelian B

$$\mu(T^{-1}(B)) = \nu(B)$$

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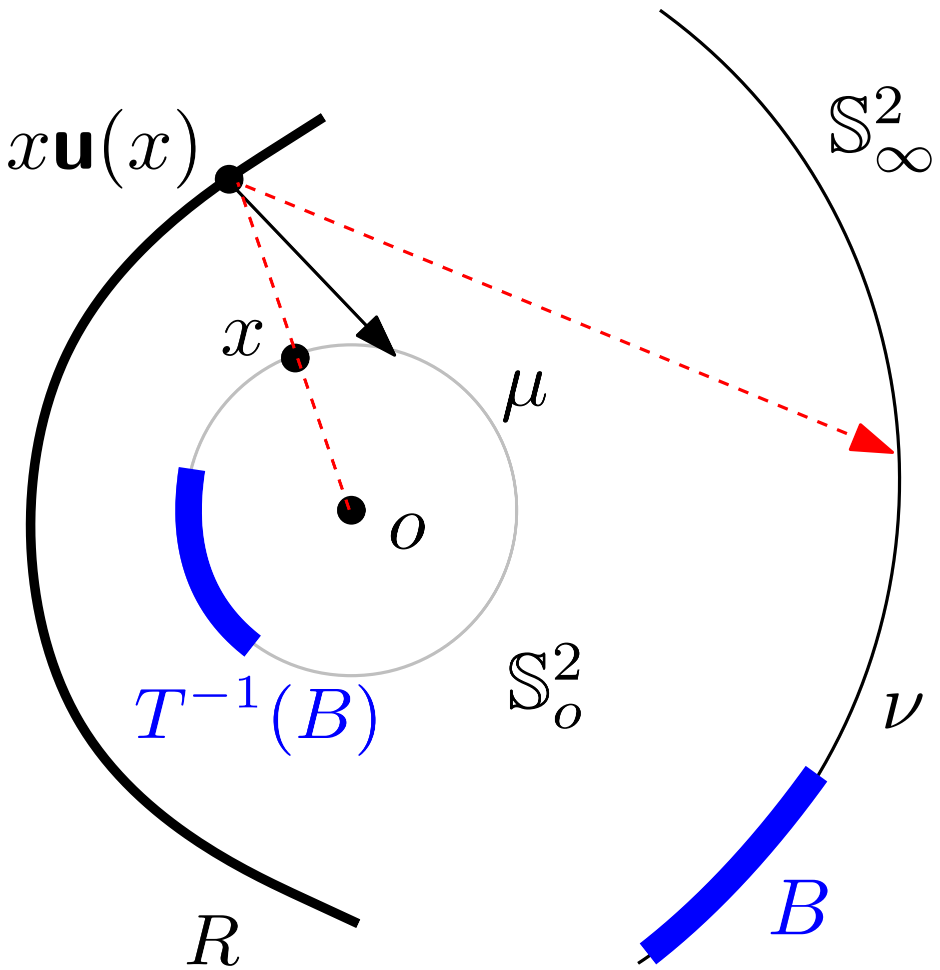

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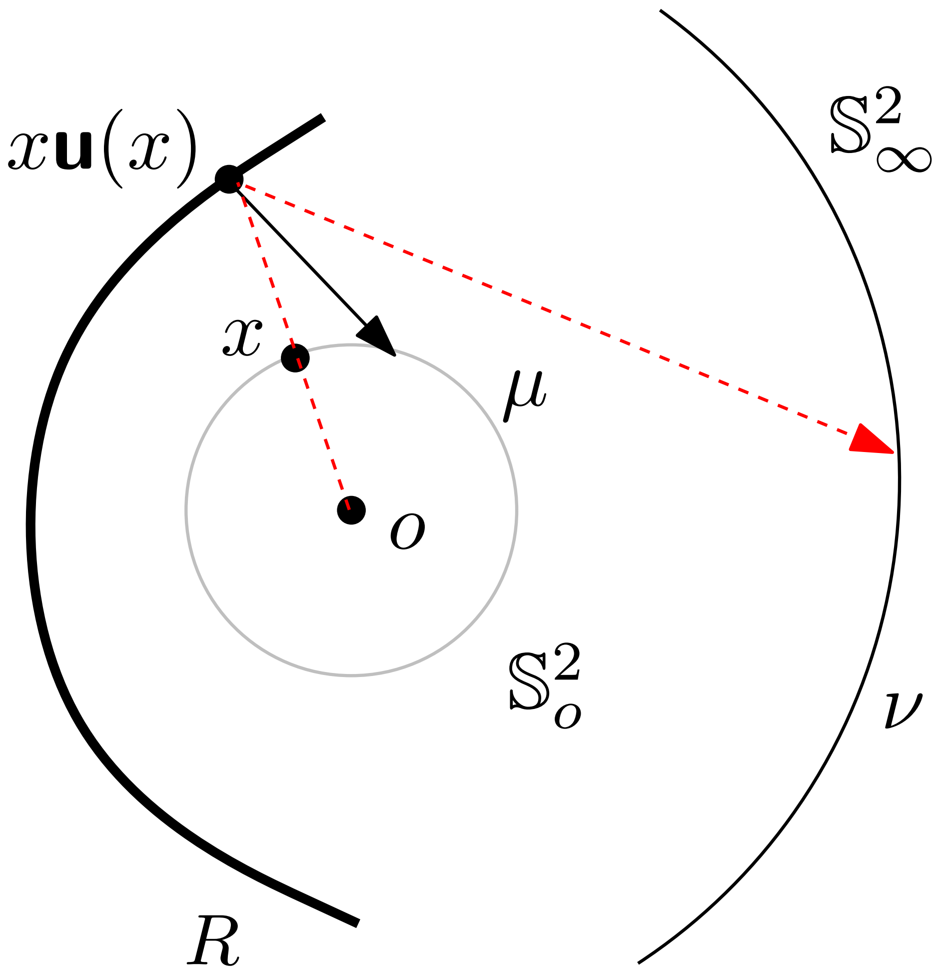
Change of variable

If $\mu(x) = f(x)dx$ and $\nu(y) = g(y)dy$

$$g(T(x)) \det(DT(x)) = f(x)$$

Mirror / Point light source



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Monge-Ampère equation Find $\mathbf{u} : \mathbb{S}^2_0 \to \mathbb{R}^+$ s.t.

$$\begin{cases} g(T_{\mathbf{u}}(x)) \det(DT_{\mathbf{u}}(x)) = f(x) \\ T_{\mathbf{u}}(x) = x - \langle x | n_{\mathbf{u}}(x) \rangle n_{\mathbf{u}}(x) \\ n_{\mathbf{u}}(x) = \frac{\nabla \mathbf{u}(x) - \mathbf{u}(x) x}{\sqrt{\|\nabla \mathbf{u}(x)\|^2 + \mathbf{u}(x)^2}} \end{cases},$$

with boundary and other conditions

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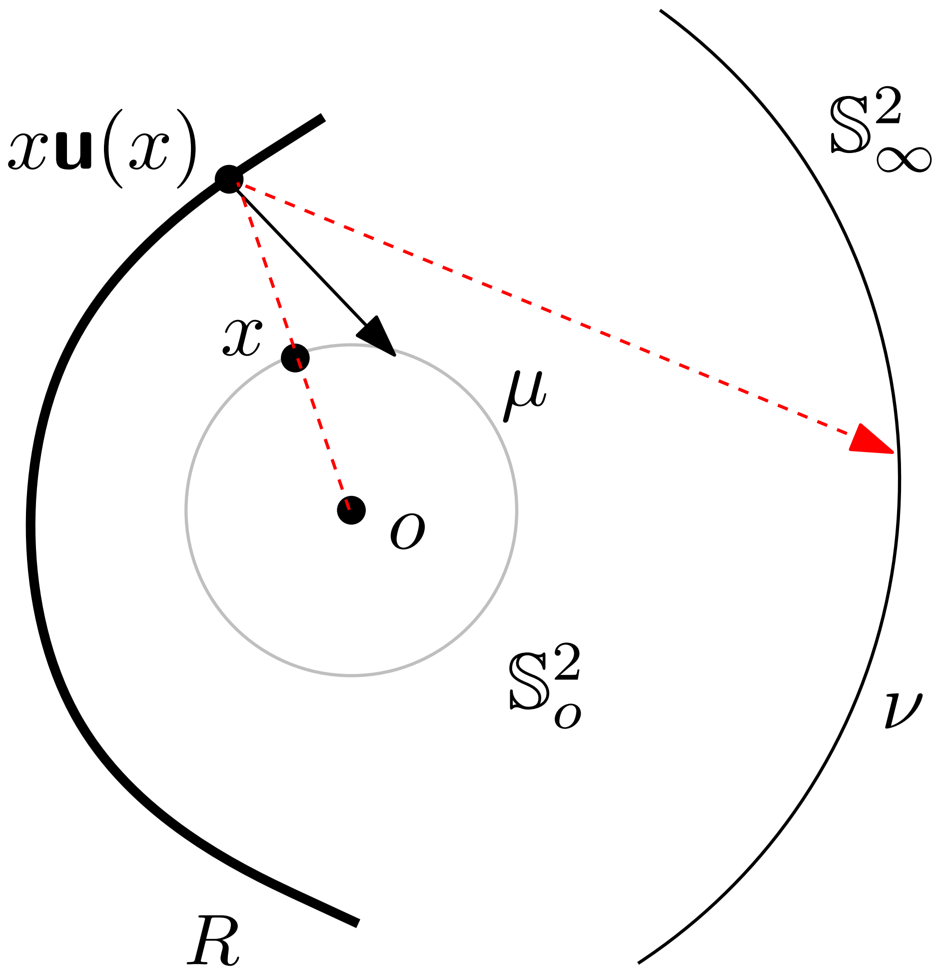

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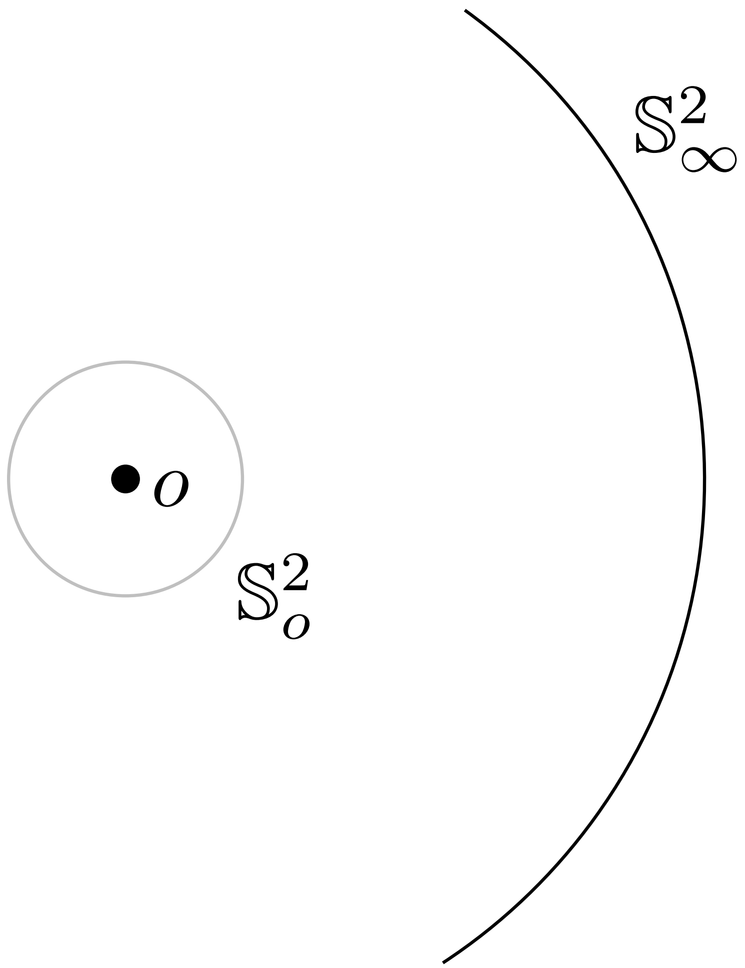
with boundary and other conditions

- Existence of weak solutions
  Caffarelli & Oliker 94
- Existence of solutions, regularity
  Wang 96, Guan & Wang 98, Caffarelli Gutierrez & Huang '08

Mirror / Point light source: semi-discrete



Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

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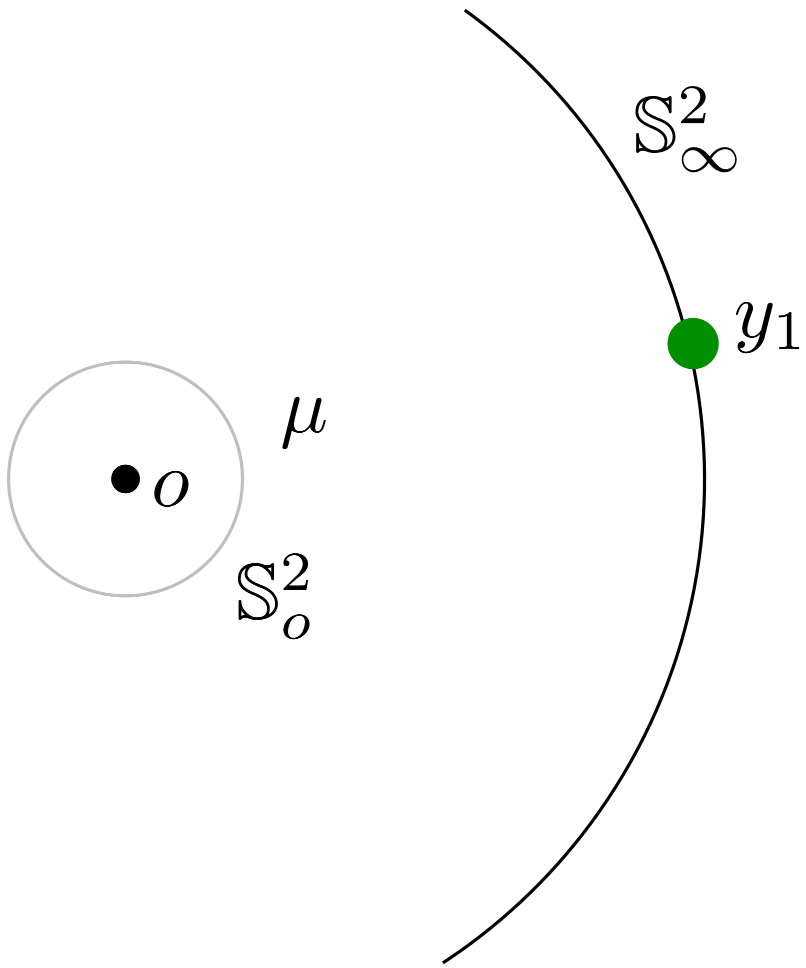
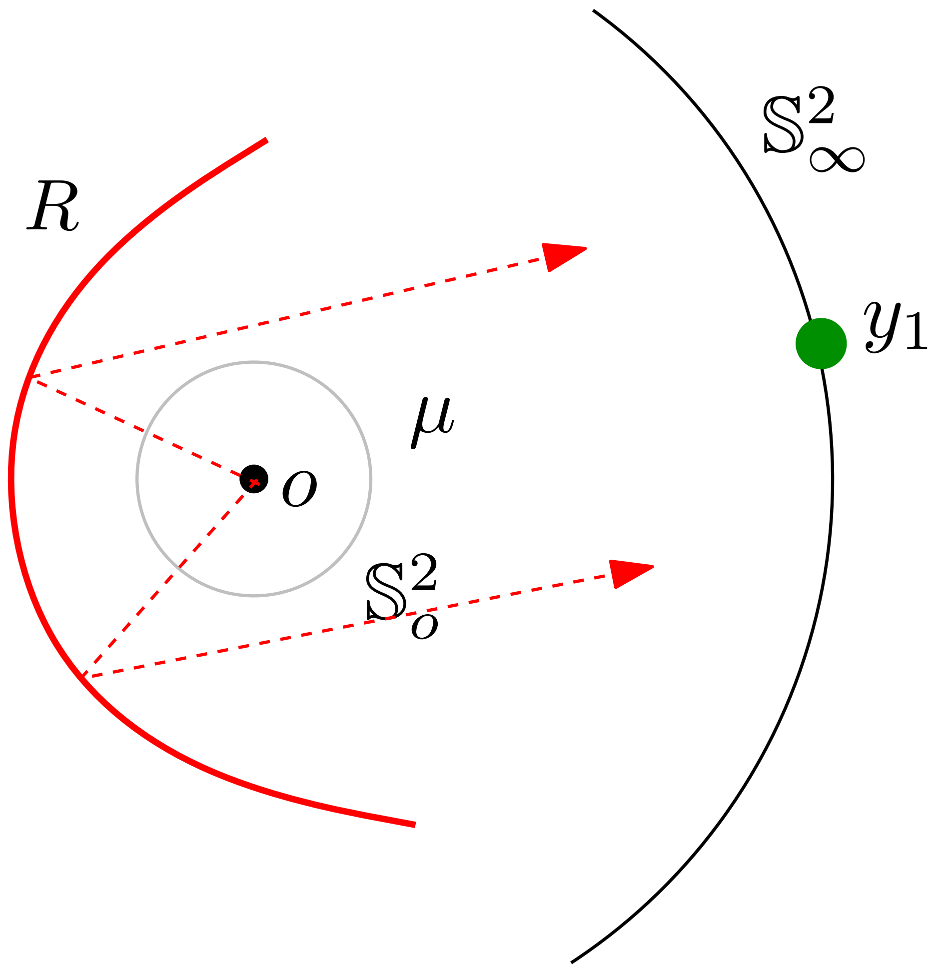


Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

Prescribed far-field: $\nu = \nu_1 \delta_{y_1}$ on $\mathbb{S}^2_\infty$

Mirror / Point light source: semi-discrete



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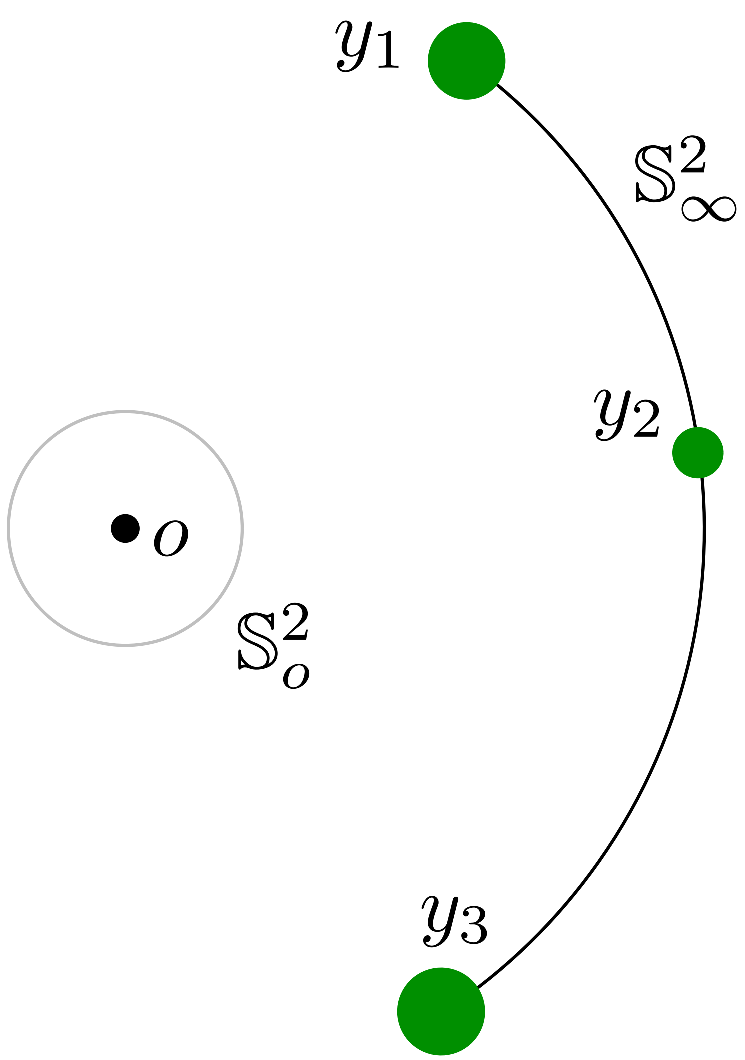
R : paraboloid of direction y_1 and focal O

Mirror / Point light source: semi-discrete



Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathcal{S}^2_\infty$

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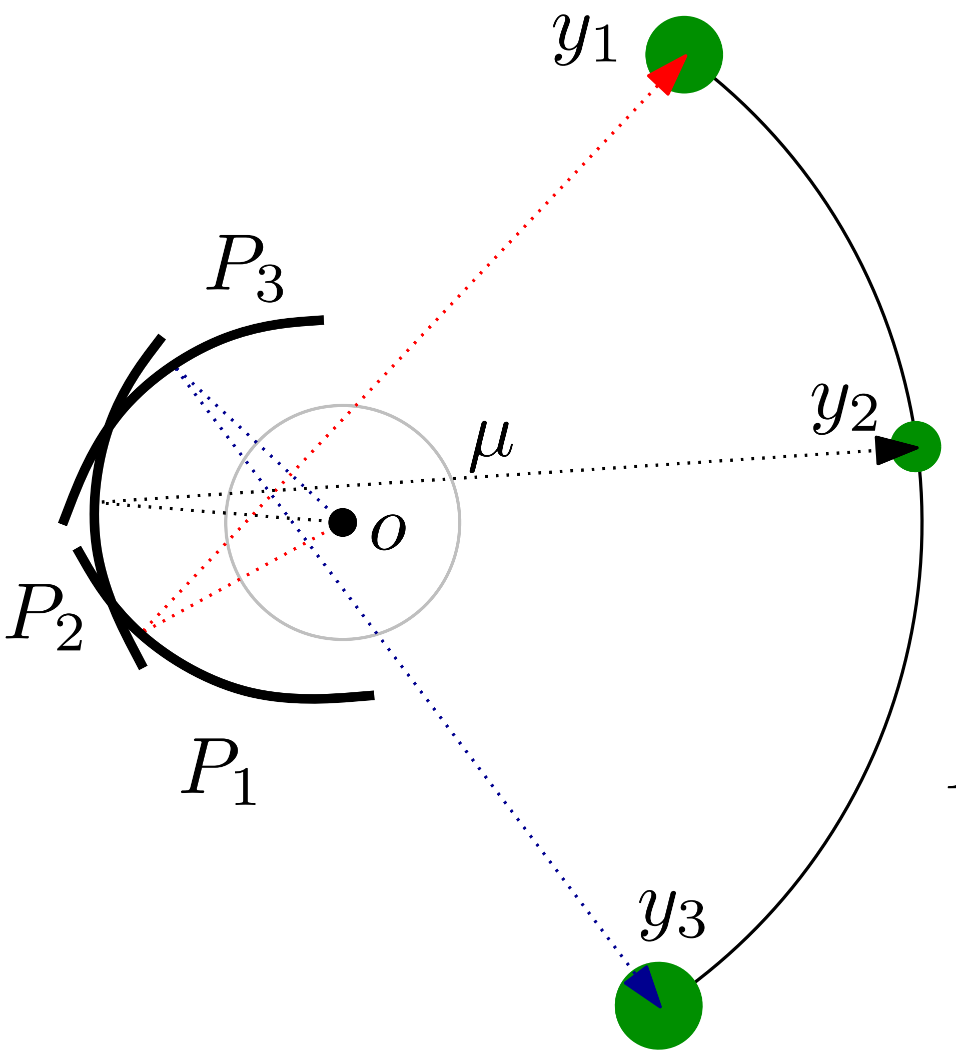

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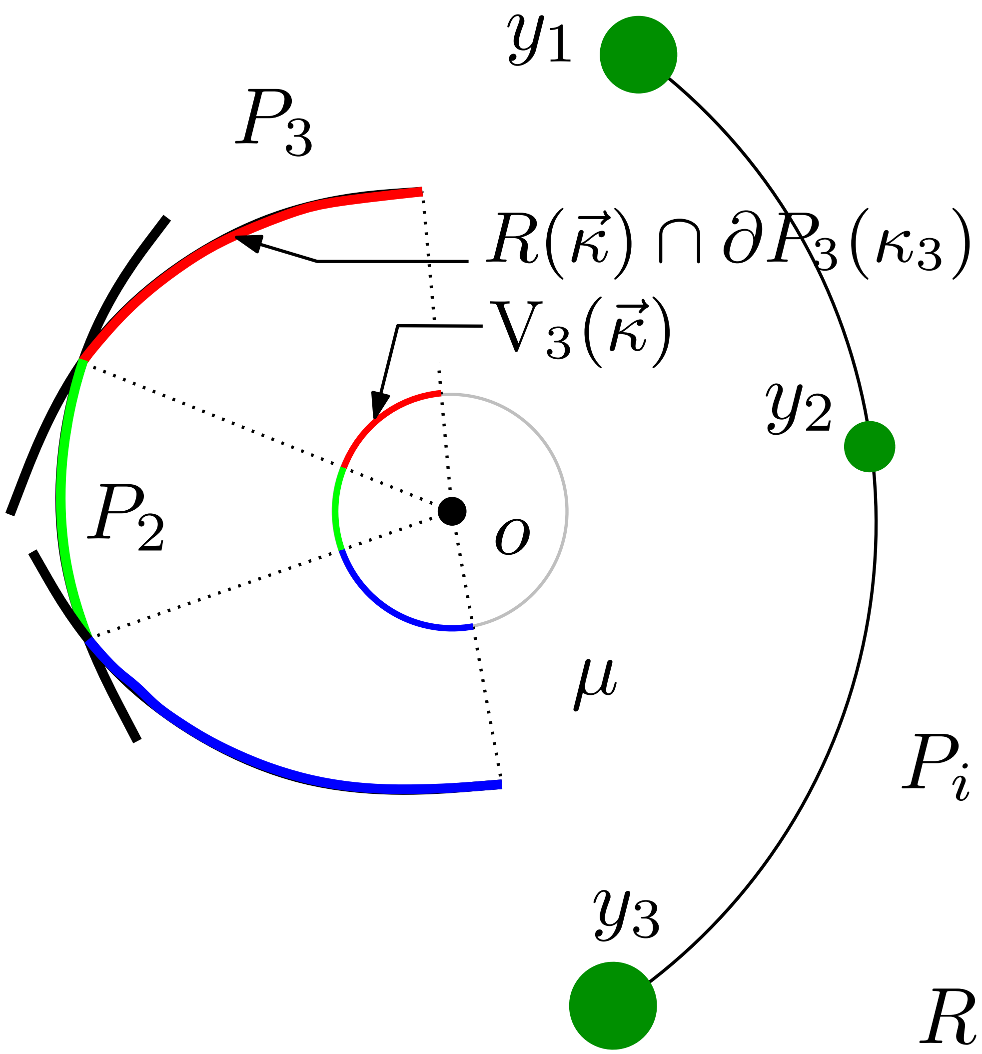
Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathcal{S}^2_\infty$

$P_i(\kappa_i) =$ solid paraboloid of revolution with focal o, direction y_i and focal distance κ_i

$R(\vec{\kappa}) = \partial \left(\cap_{i=1}^N P_i(\kappa_i) \right)$

Mirror / Point light source: semi-discrete



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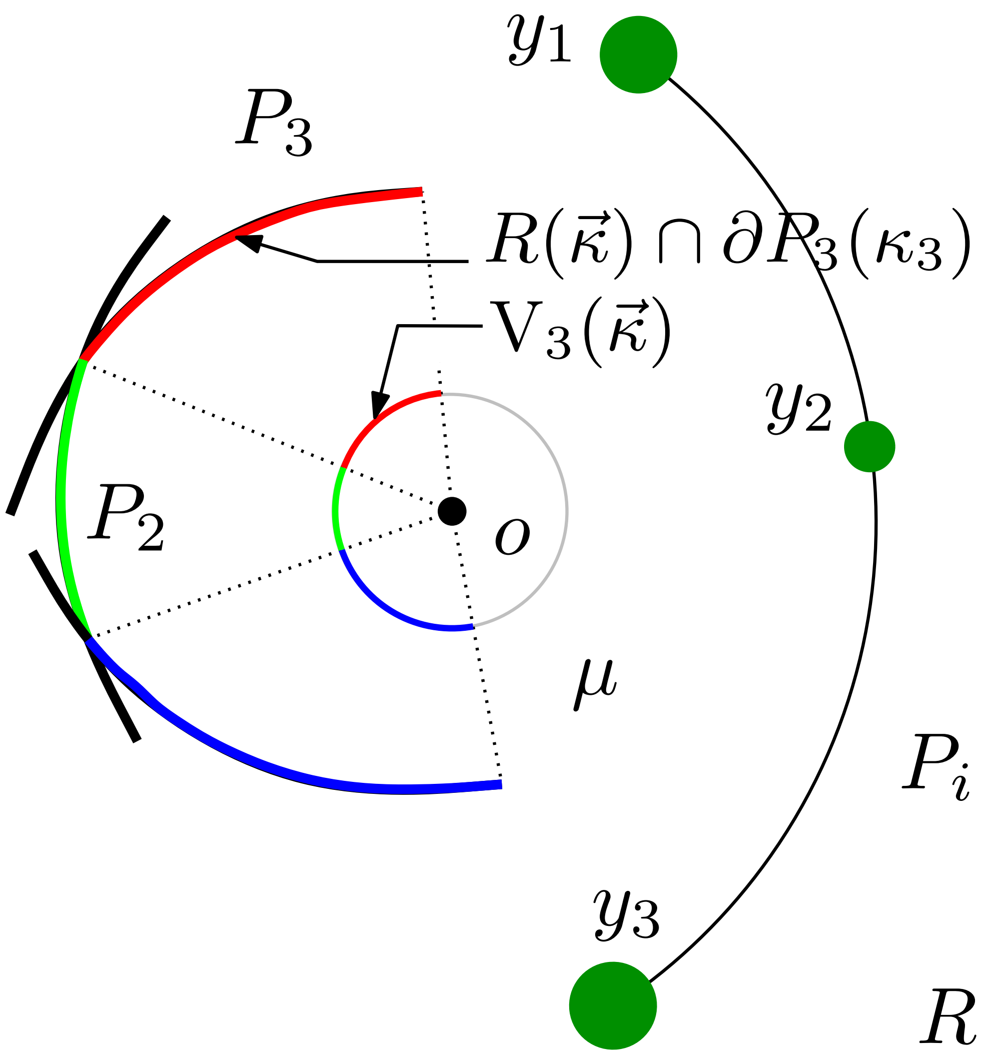
Decomposition of $\mathbb{S}^2_o$: $\mathrm{V}_i(\vec{\kappa}) = \pi_{\mathcal{S}^2_o}(R(\vec{\kappa}) \cap \partial P_i(\kappa_i))$
$=$ directions that are reflected towards y_i.

Mirror / Point light source: semi-discrete



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$=$ directions that are reflected towards y_i.

Problem (FF): Find $\kappa_1, \ldots, \kappa_N$ such that for every i, $\boxed{\mu(V_i(\vec{\kappa}))} = \nu_i$.

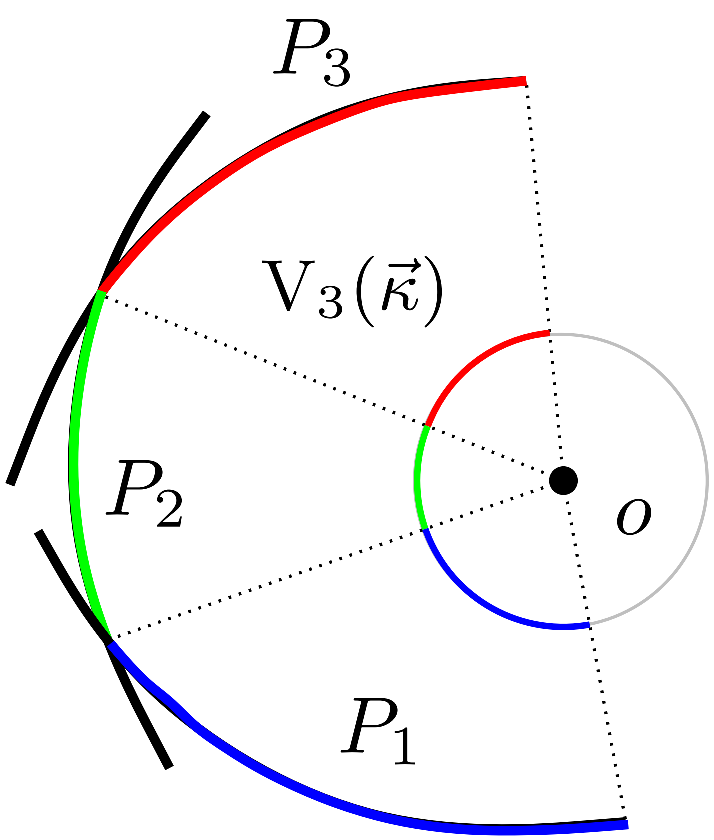
amount of light reflected in direction y_i.

Mirror / Point light source: Optimal Transport

Lemma: With $c(x, y) = -\log(1 - \langle x|y\rangle)$, and $\psi_i := \log(\kappa_i)$,

$$\mathrm{V}_i(\vec{\kappa}) = \{x \in \mathbb{S}^2_0,\ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j\}.$$

Caffarelli-Oliker '94



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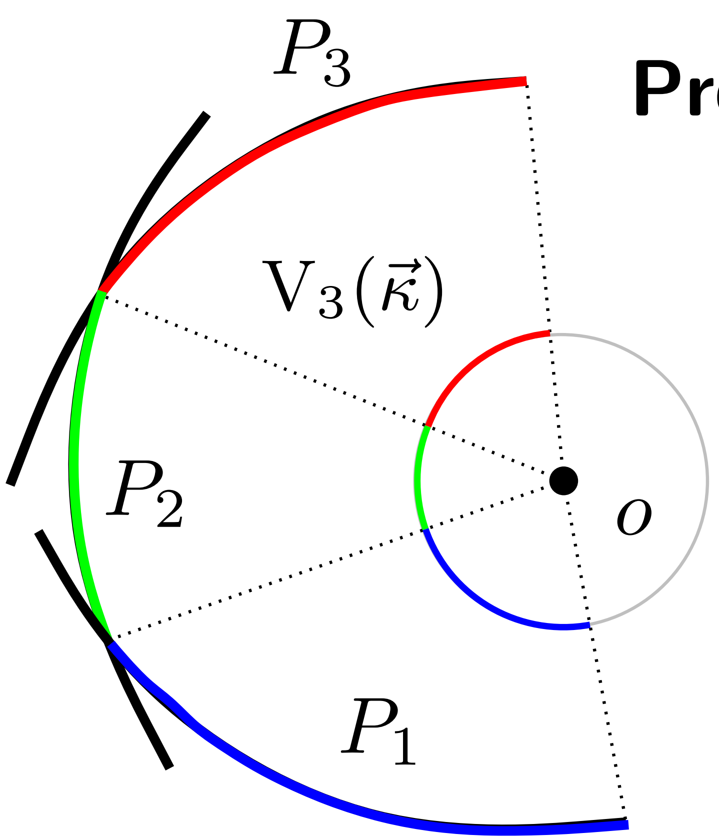

Proof: $\partial P_i(\kappa_i)$ is parameterized in radial coordinates by
$$\rho_i : x \in \mathbb{S}^2_o \mapsto \frac{\kappa_i}{1 - \langle x|y_i\rangle}$$

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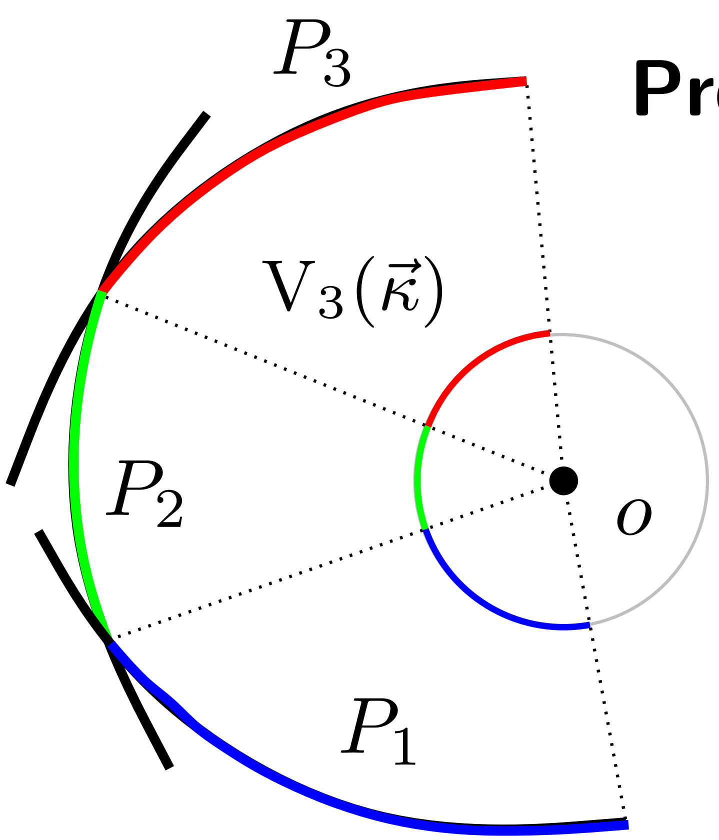

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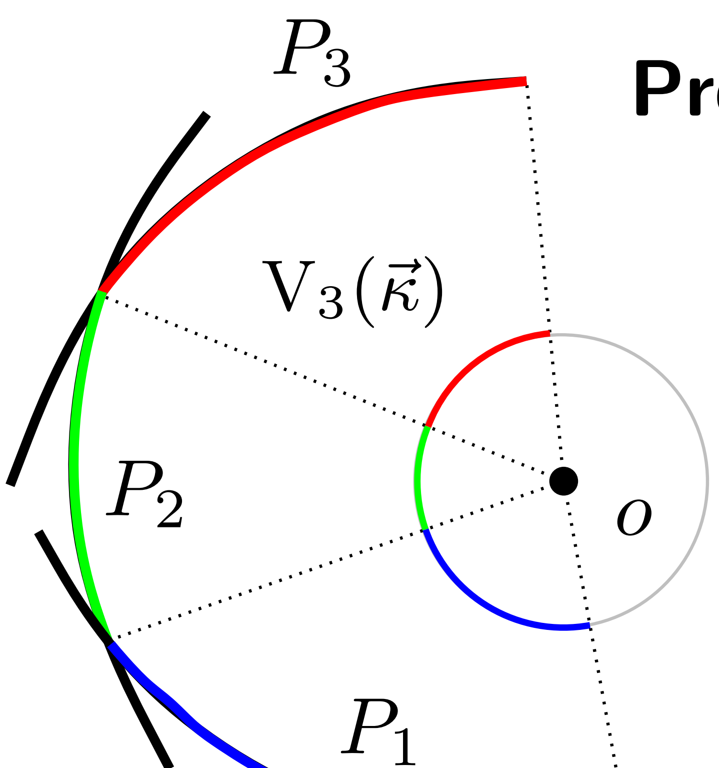

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$$\iff \log(\kappa_i) - \log(1 - \langle x|y_i\rangle) \leq \cdots$$

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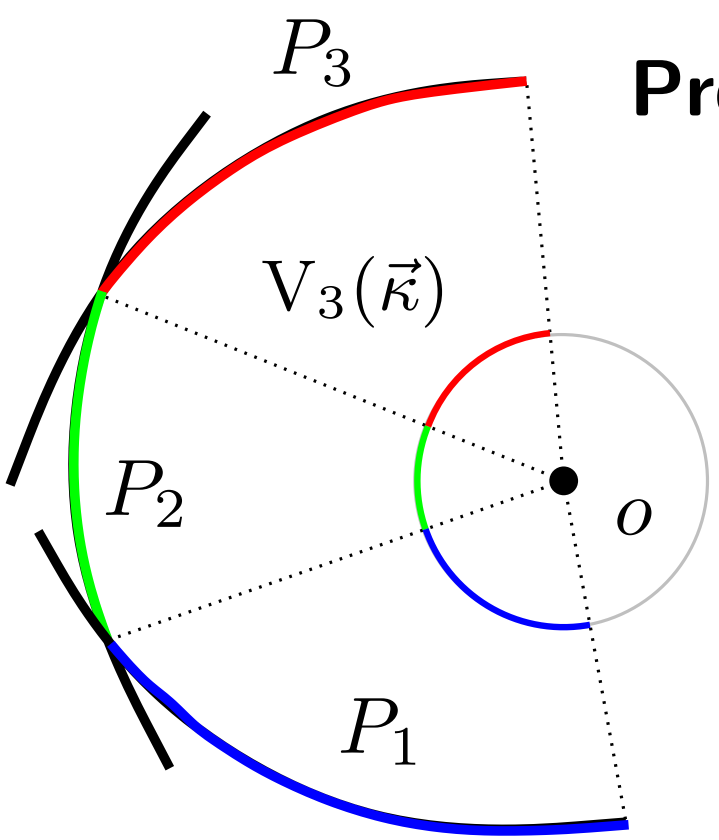

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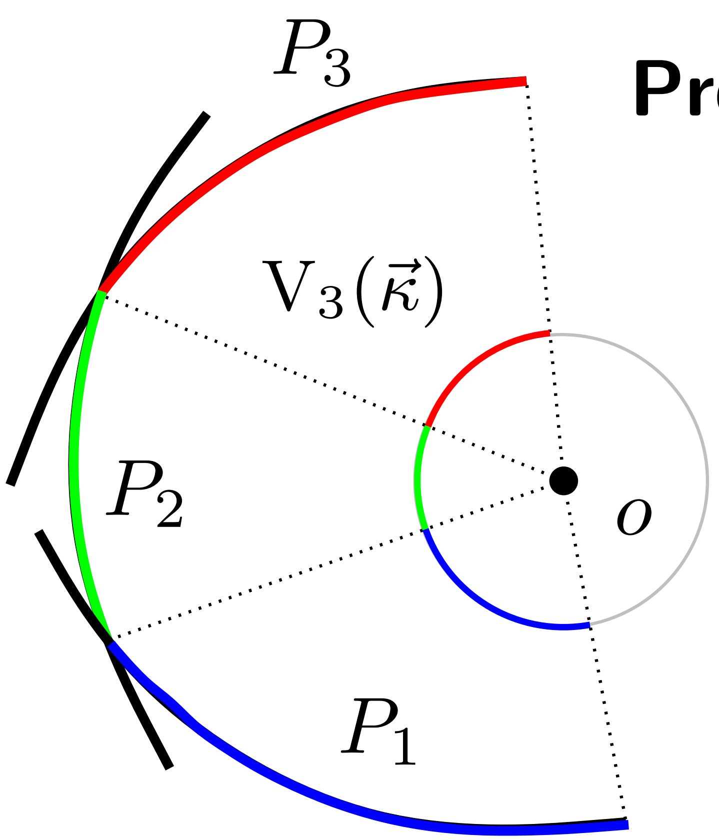
$$\begin{aligned} x \in \mathrm{V}_i(\vec{\kappa}) \quad & \Longleftrightarrow \frac{\kappa_i}{1-\langle x|y_i\rangle} \leq \frac{\kappa_j}{1-\langle x|y_j\rangle} \\ & \Longleftrightarrow \log(\kappa_i) - \log(1 - \langle x|y_i\rangle) \leq \cdots \\ & \Longleftrightarrow \psi_i + c(x,y_i) \leq \psi_j + c(x,y_j) \end{aligned}$$

Mirror / Point light source: Optimal Transport

Lemma: With $c(x,y) = -\log(1 - \langle x|y\rangle)$, and $\psi_i := \log(\kappa_i)$,
$$\mathrm{V}_i(\vec{\kappa}) = \{x \in \mathbb{S}^2_0,\ c(x,y_i) + \psi_i \leq c(x,y_j) + \psi_j \quad \forall j\}.$$

Caffarelli-Oliker '94



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$\rightsquigarrow$ An optimal transport problem on $\mathbb{S}^2$ Wang '04

Problem (FF): Find $\kappa_1, \ldots, \kappa_N$ such that for every i, $\mu(V_i(\vec{\kappa})) = \nu_i$.

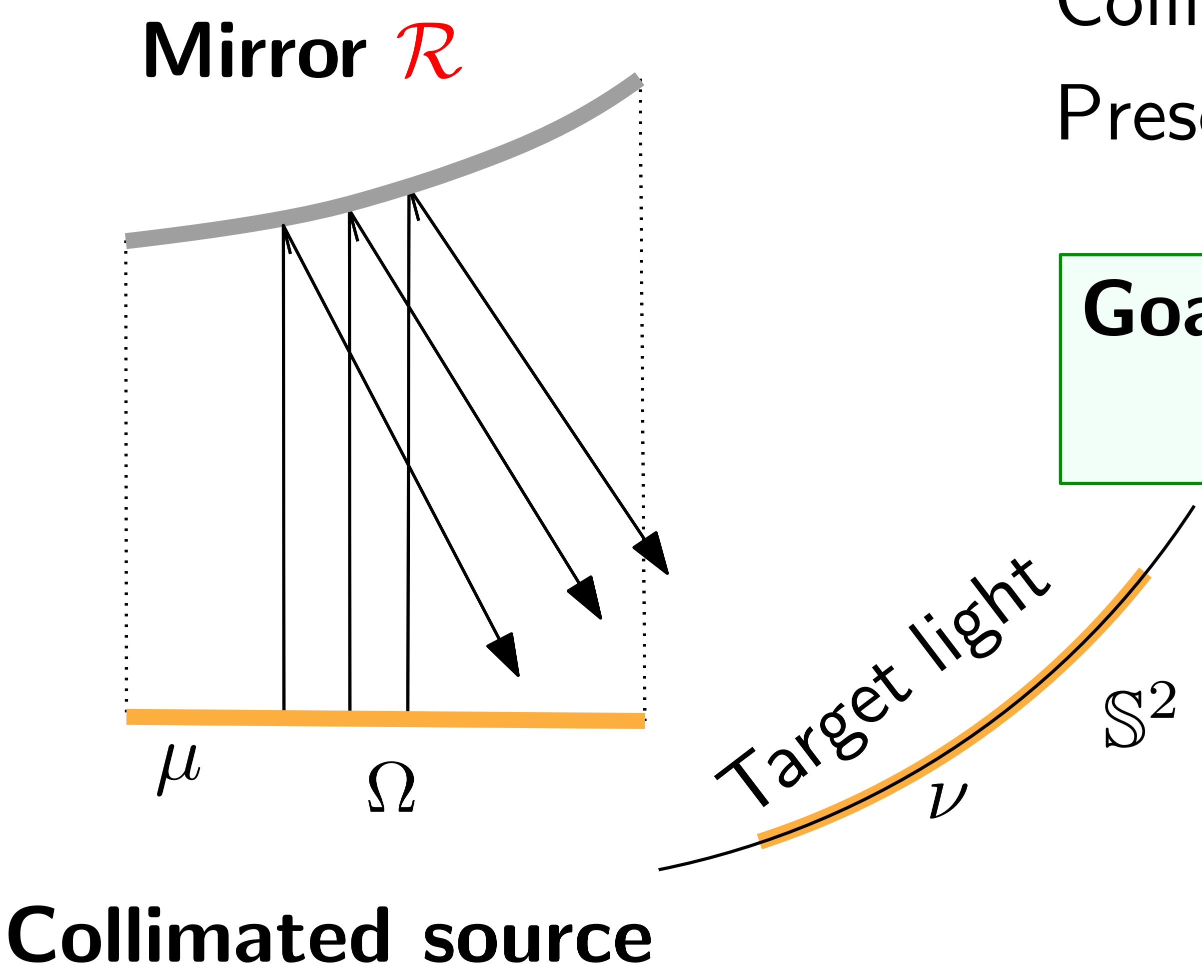
Outline

- Case 1: mirror for point light source
- Case 2: mirror for collimated light source

- Optimal transport
- Semi-discrete optimal transport
- Damped Newton algorithm

- Non-imaging optics: Far-Field target
- Non-imaging optics: Near-Field target

Mirror / Collimated light source

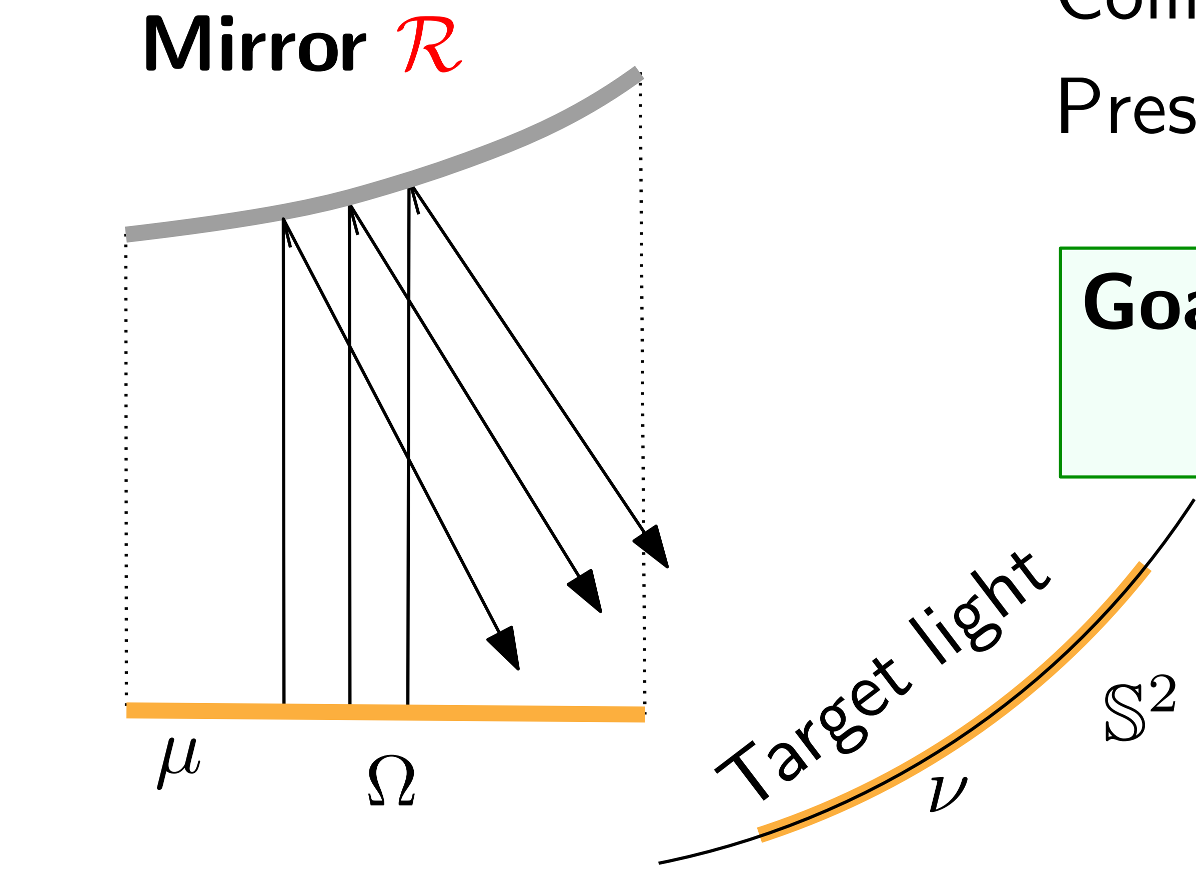


Collimated light μ measure on $\Omega \subset \mathbb{R}^2 \times \{0\}$

Prescribed far-field: ν on $\mathbb{S}^2$

Goal: Find a surface R which sends (Ω, μ) to $(\mathbb{S}^2, \nu)$ under reflection by Snell's law.

Mirror / Collimated light source



Collimated source

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- R param. by $x \in \Omega \mapsto (x, \mathbf{u}(x))$ where $\mathbf{u} : \Omega \to \mathbb{R}$ height function
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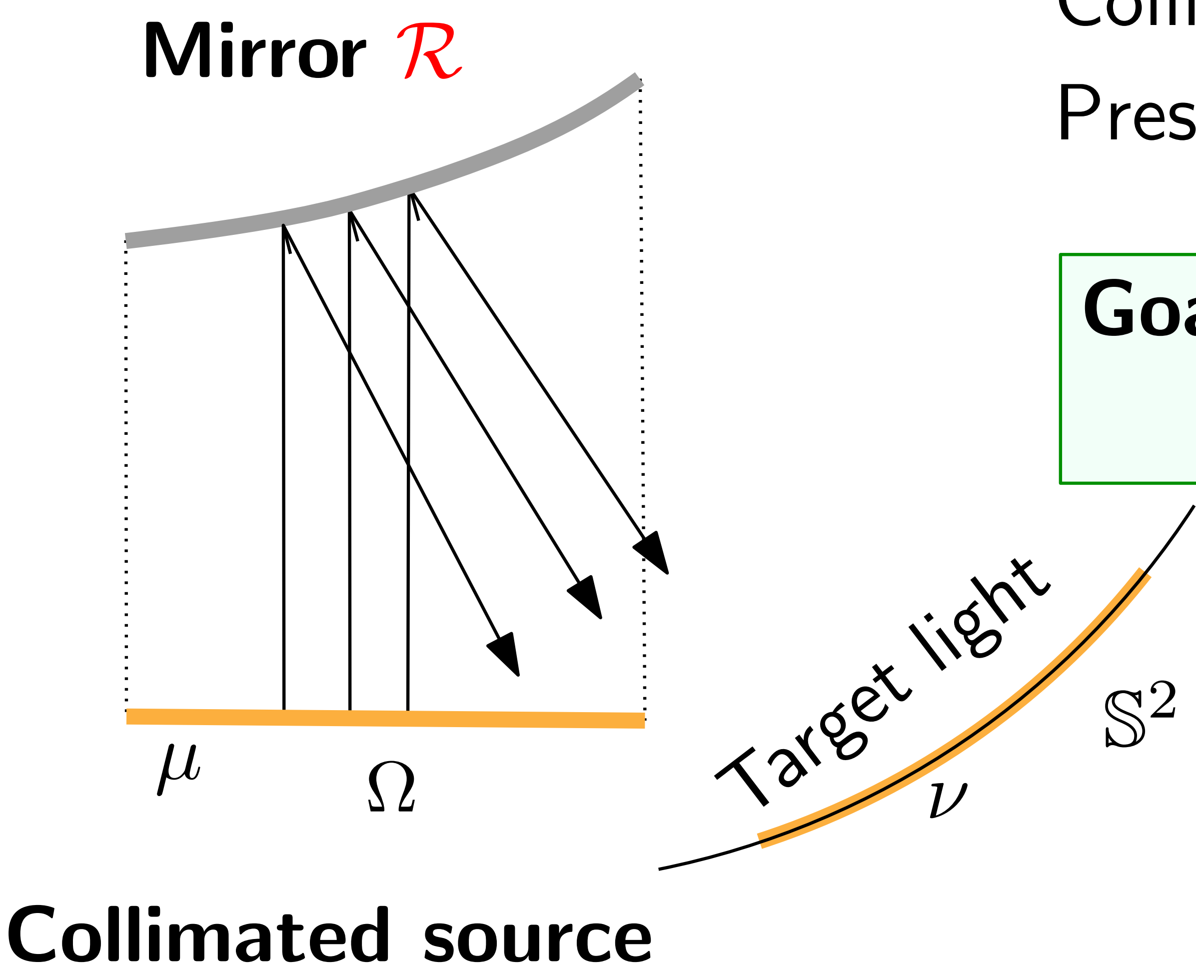
Mirror / Collimated light source



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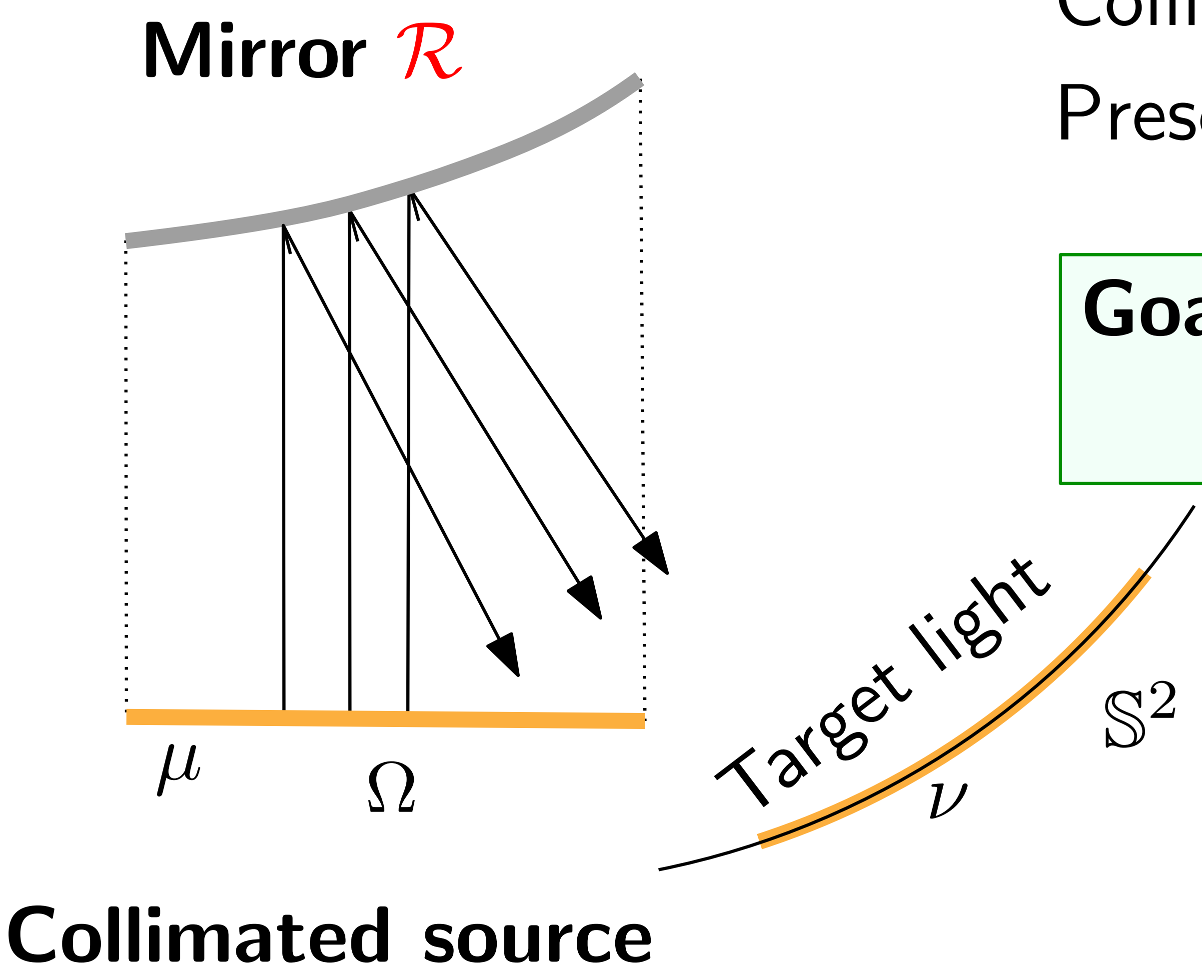
Brenier formulation $(F \circ \nabla \mathbf{u})_\sharp \mu = \nu$

$$\Leftrightarrow \forall A \ \mu((F \circ \nabla \mathbf{u})^{-1}(A)) = \nu(A)$$

Mirror / Collimated light source



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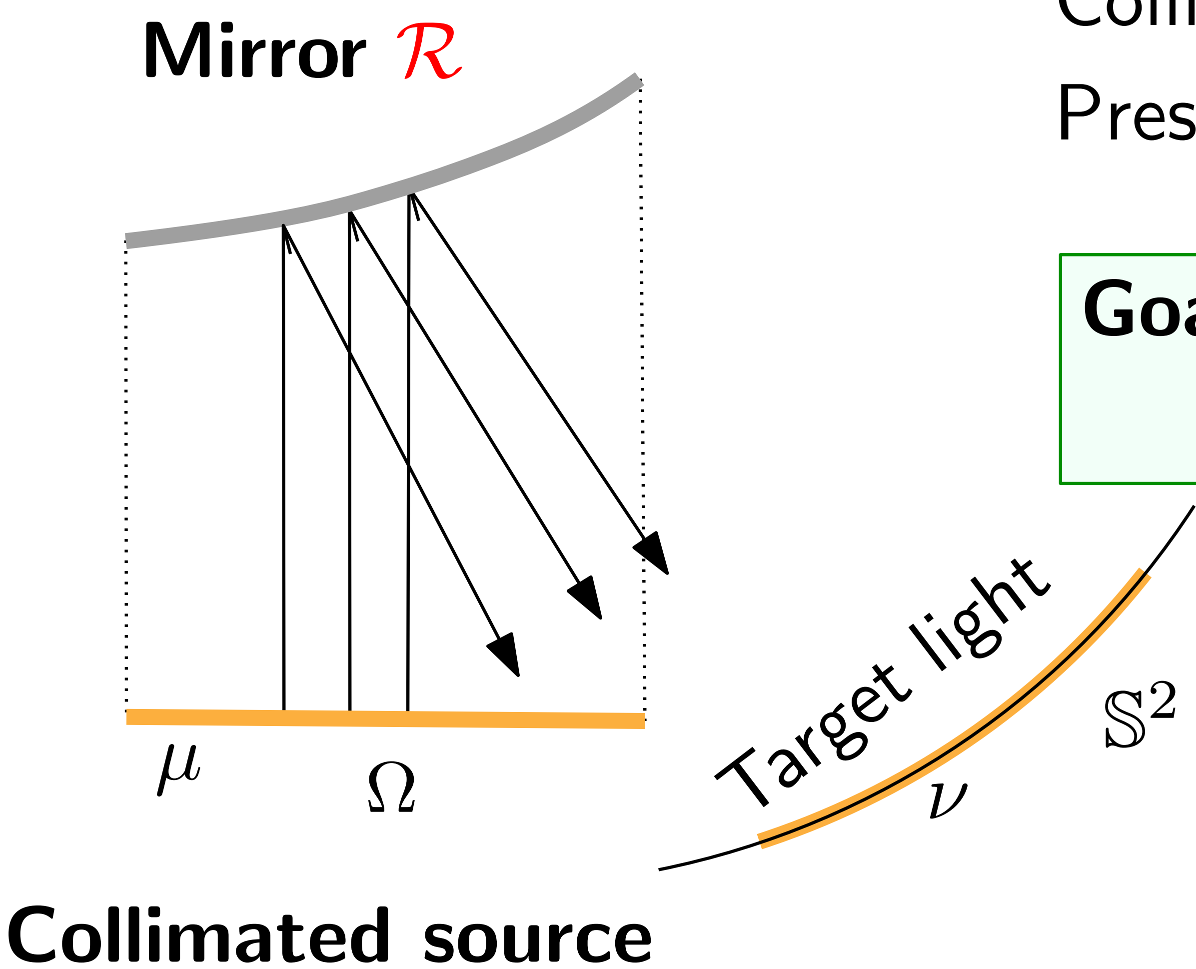
$$\Leftrightarrow \forall A \;\; \mu((F \circ \nabla \mathbf{u})^{-1}(A)) = \nu(A)$$

$$\Leftrightarrow \forall B \;\; \mu((\nabla \mathbf{u})^{-1}(B)) = \tilde{\nu}(B) \quad \text{with } B = F^{-1}(A) \subset \mathbb{R}^2$$

Mirror / Collimated light source



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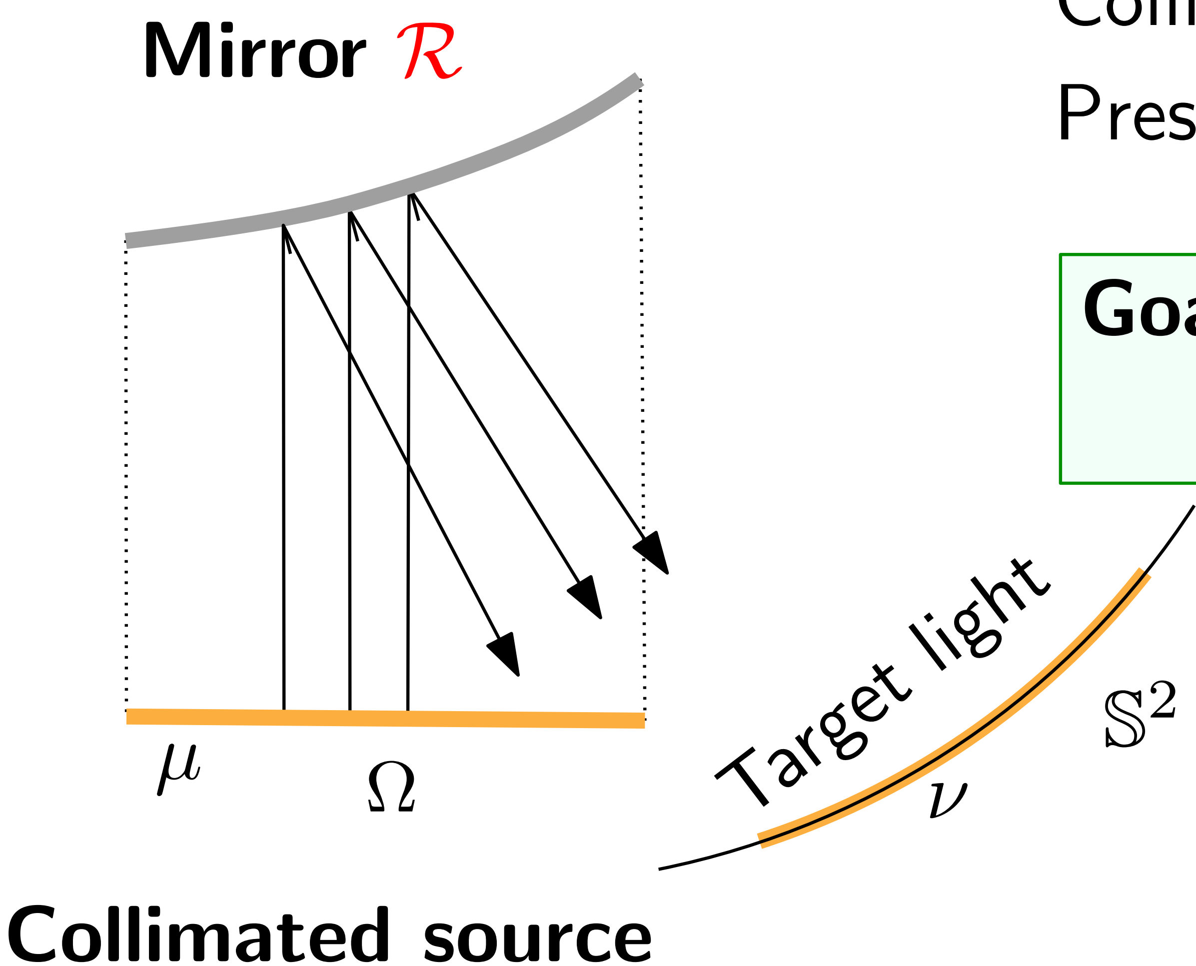
$\Leftrightarrow \forall B \;\; \mu((\nabla \mathbf{u})^{-1}(B)) = \tilde{\nu}(B)$ with $B = F^{-1}(A) \subset \mathbb{R}^2$

$\Leftrightarrow \det(\nabla^2 \mathbf{u}(x)) g(\nabla \mathbf{u}(x)) = f(x)$ if $\mu(x) = f(x)dx$ and $\tilde{\nu}(x) = g(x)dx$

Mirror / Collimated light source



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Monge-Ampère equation in $\mathbb{R}^2$

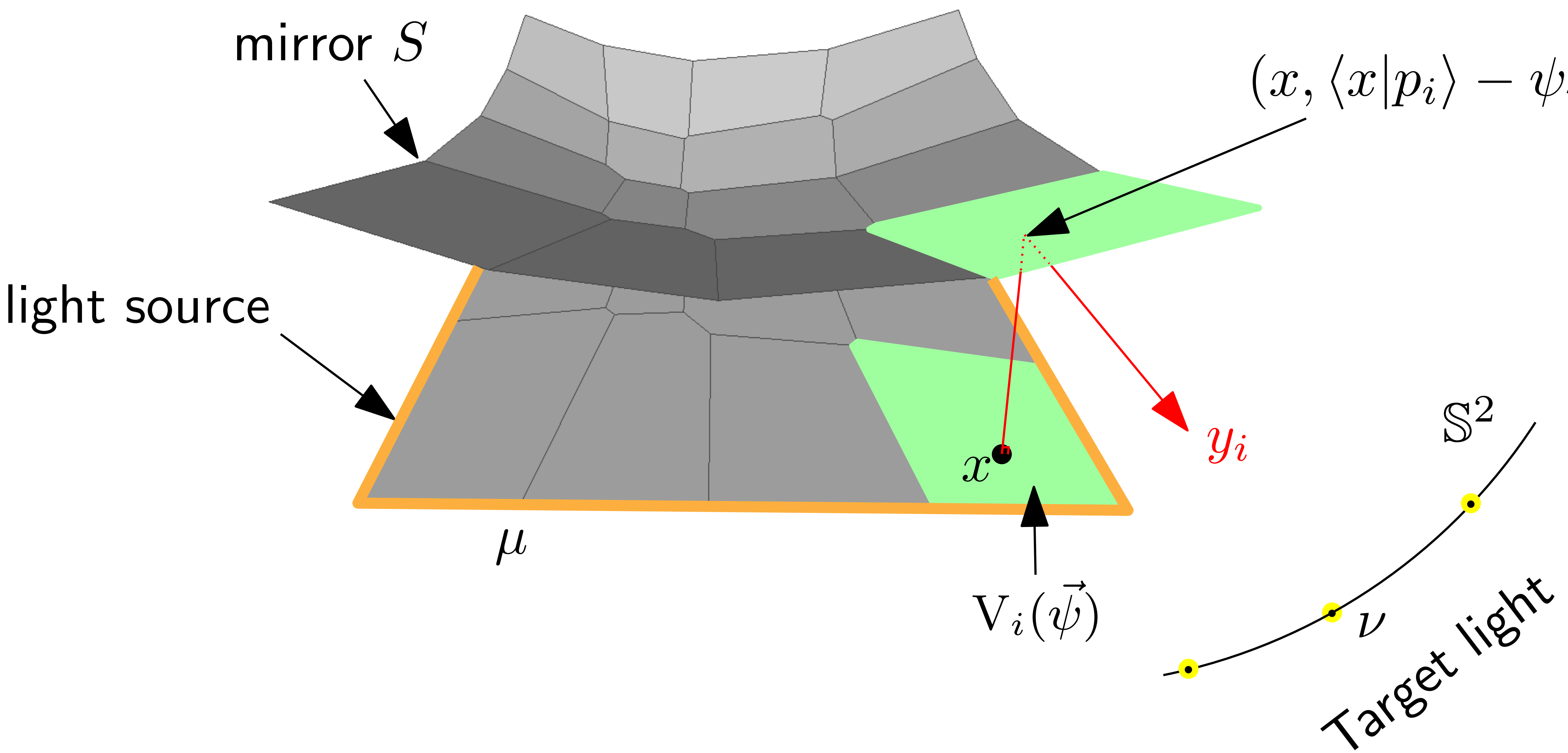
Find $\mathbf{u} : \Omega \to \mathbb{R}^2$ such that $\det(\nabla^2 \mathbf{u}(x)) g(\nabla \mathbf{u}(x)) = f(x)$

with boundary conditions

Mirror / Collimated light source: semi-discrete

Collimated light μ measure on $\Omega \subset \mathbb{R}^2 \times \{0\}$

Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{S}^2$



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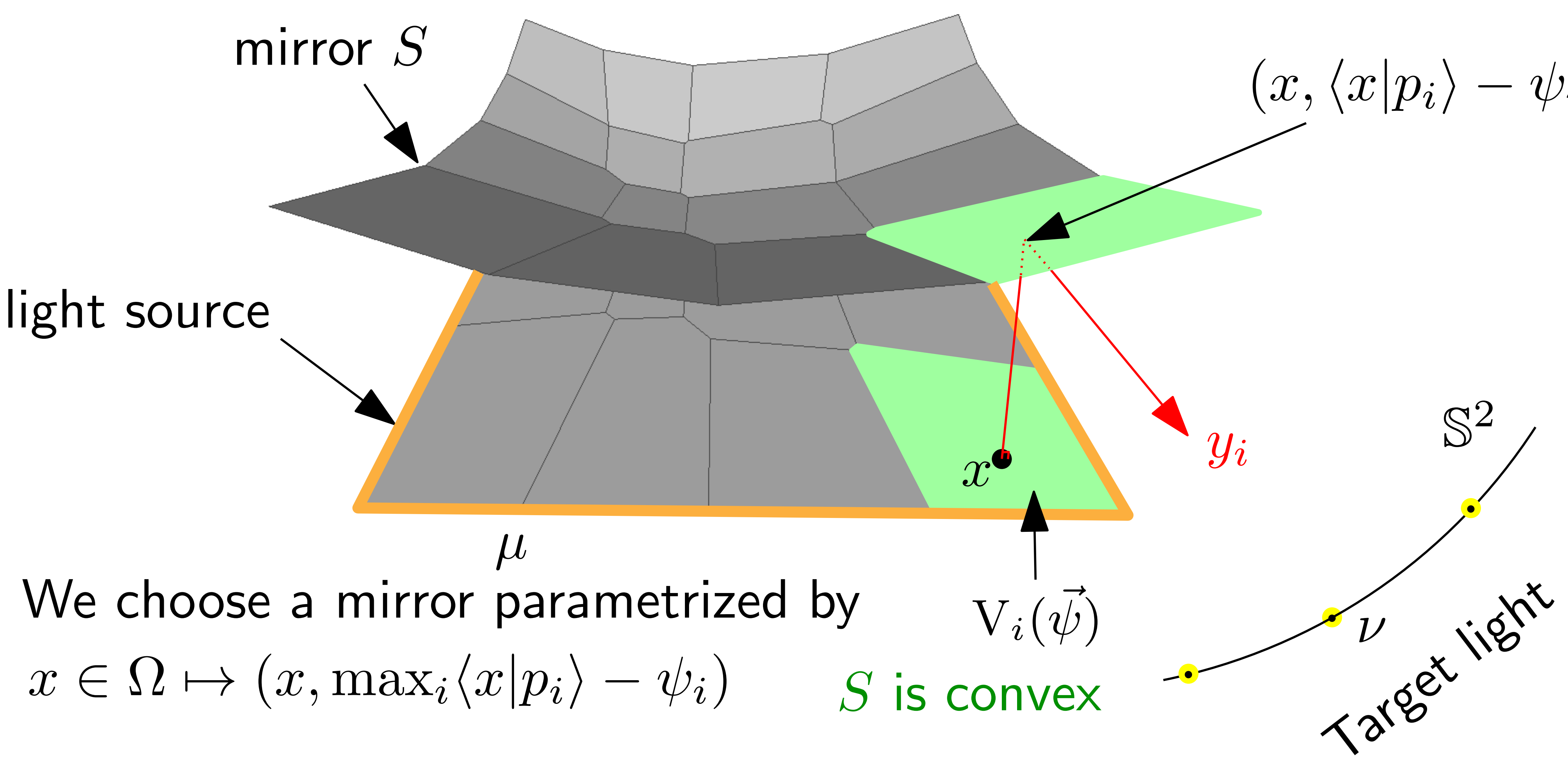

We choose a mirror parametrized by
$x \in \Omega \mapsto (x, \max_i \langle x|p_i\rangle - \psi_i)$

S is convex

Mirror / Collimated light source: semi-discrete

Collimated light μ measure on $\Omega \subset \mathbb{R}^2 \times \{0\}$

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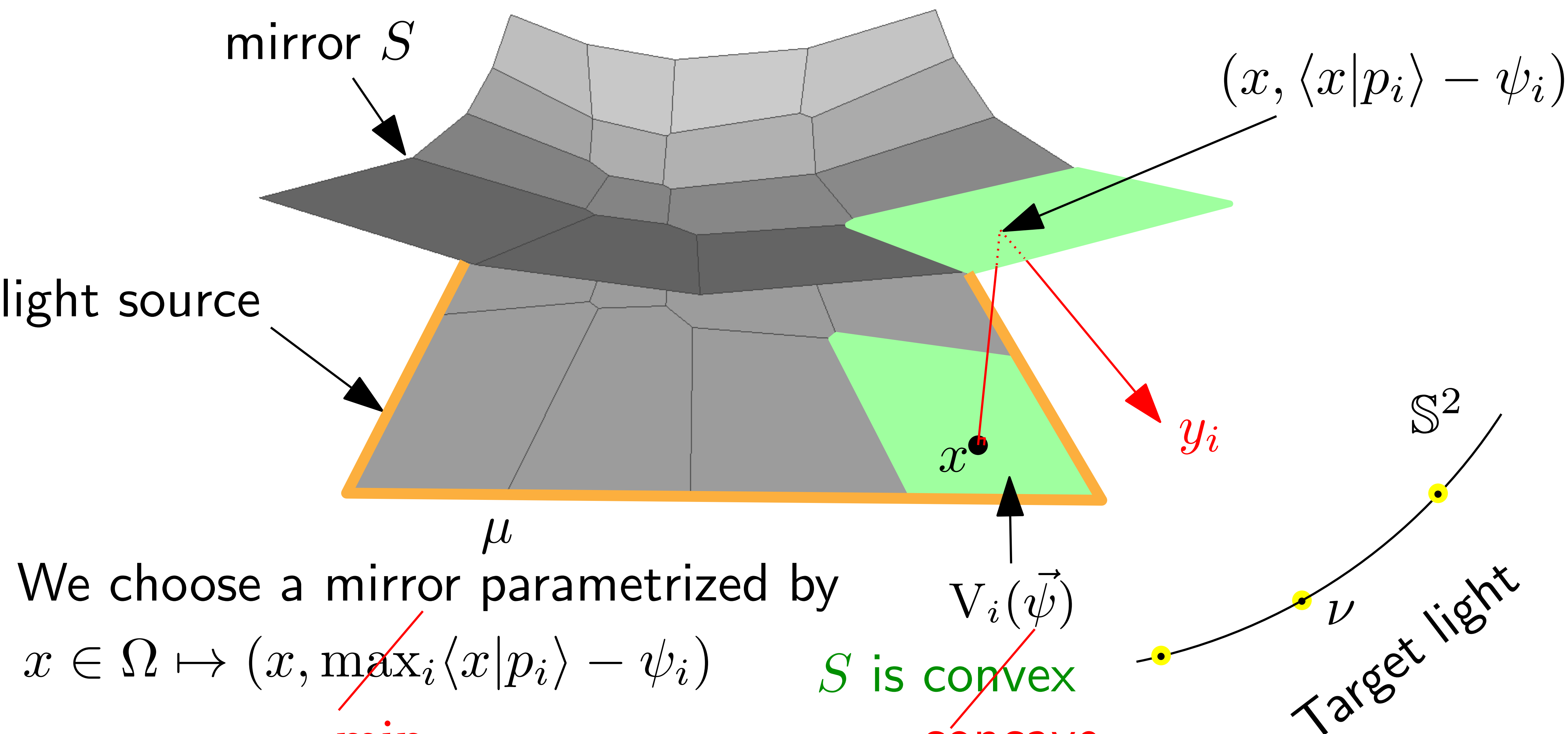

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$x \in \Omega \mapsto (x, \cancel{\max}_i \langle x|p_i\rangle - \psi_i)$

min

S is ~~convex~~

concave

Mirror / Collimated light source: semi-discrete

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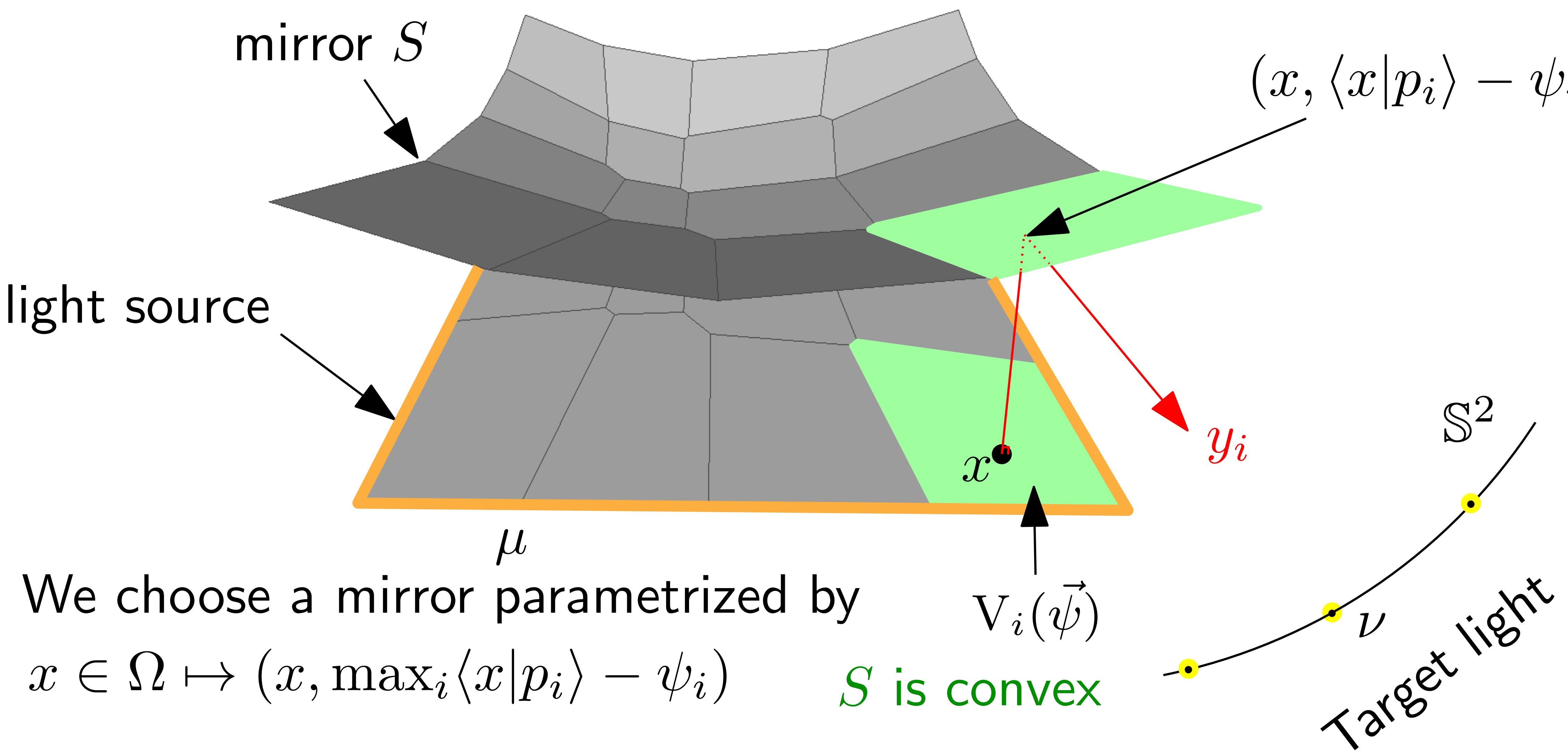

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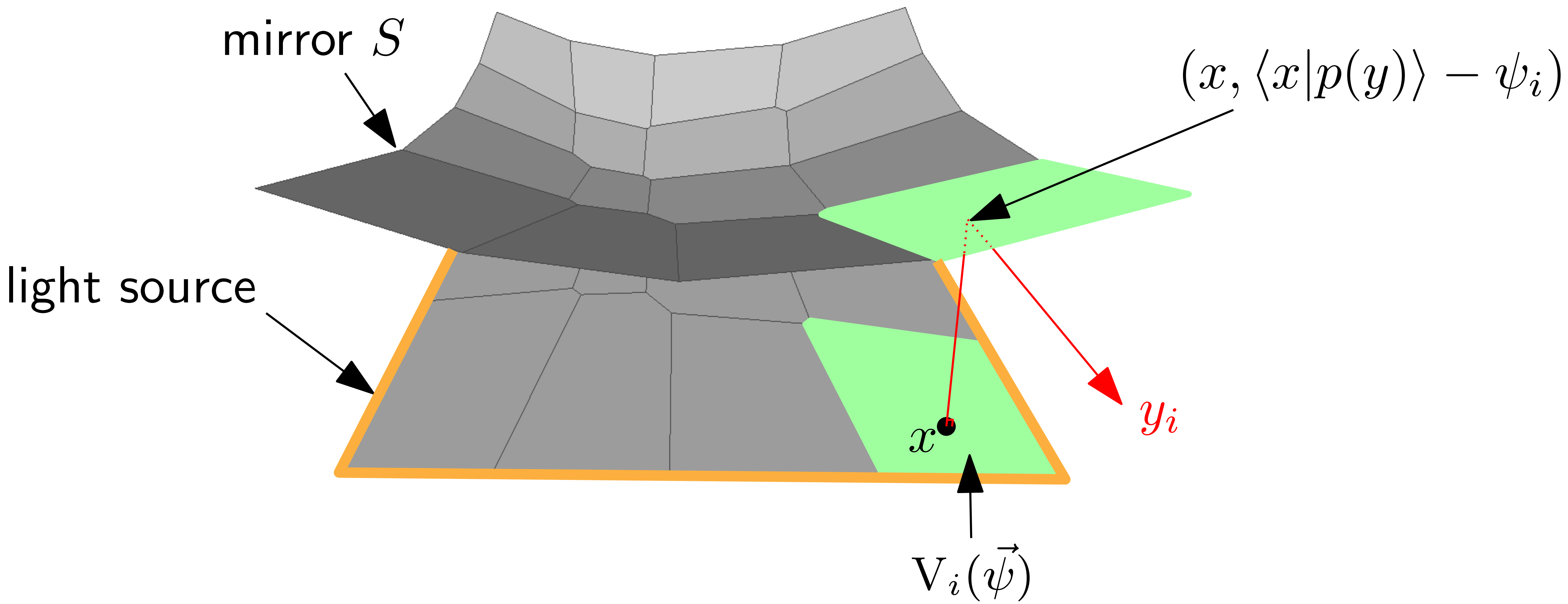
Problem (FF): Find $\psi_1, \ldots, \psi_N$ such that for every i, $\boxed{\mu(V_i(\vec{\psi}))} = \nu_i$.

amount of light reflected in direction y_i.

Mirror / Collimated source: Optimal Transport

Lemma: With $c(x,y) = -\langle x|y\rangle$

$$\mathrm{V}_i(\vec{\psi}) = \{x \in \mathbb{R}^2,\ c(x,y_i) + \psi_i \leq c(x,y_j) + \psi_j \quad \forall j\}.$$



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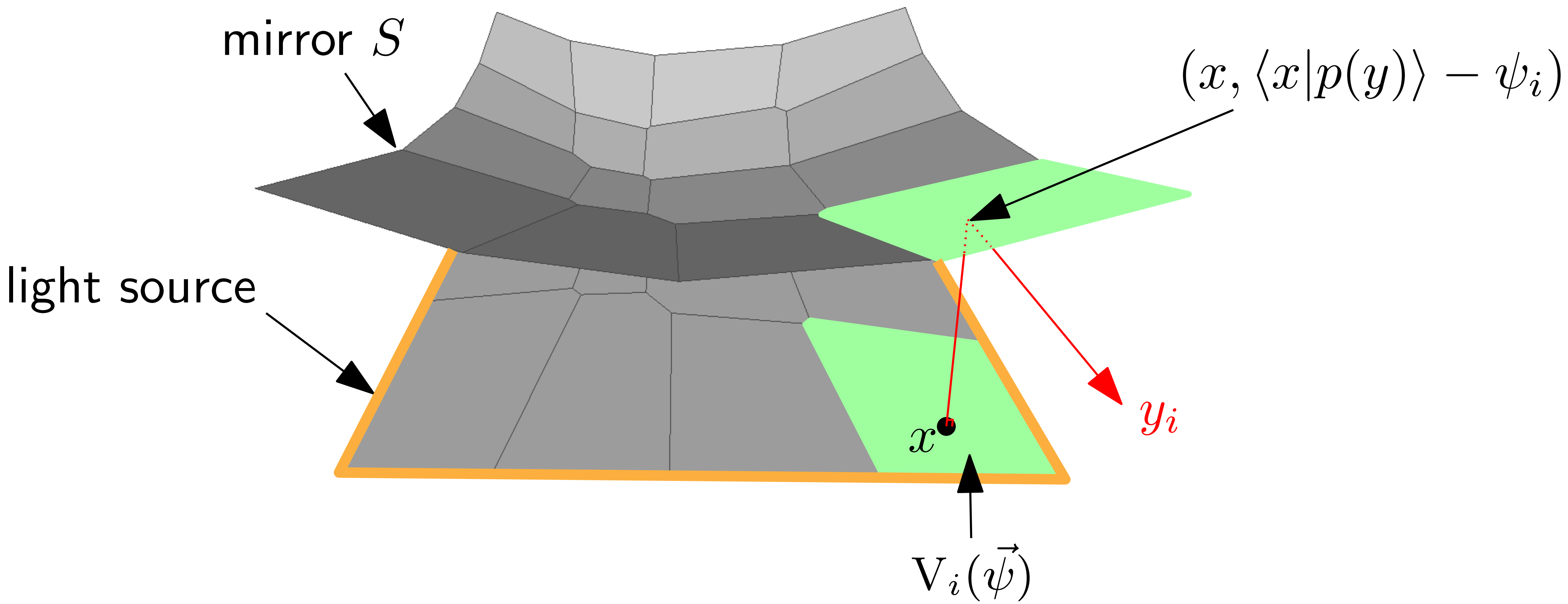

$\rightsquigarrow$ Optimal transport problem in $\mathbb{R}^2$

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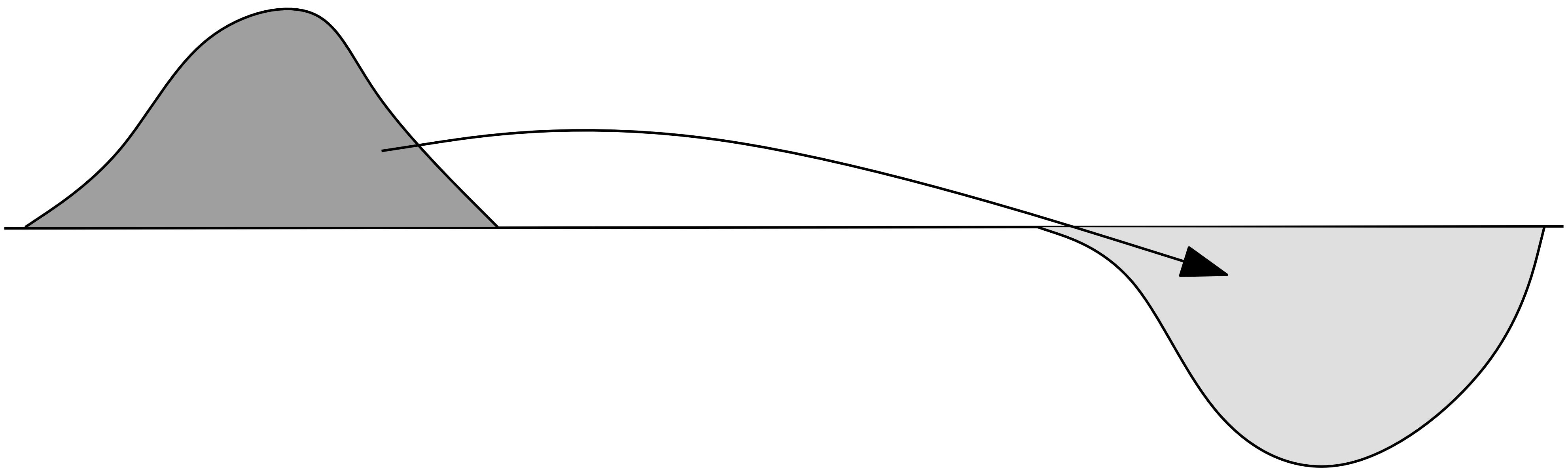
Monge problem (1781)

How to optimally move sand ?



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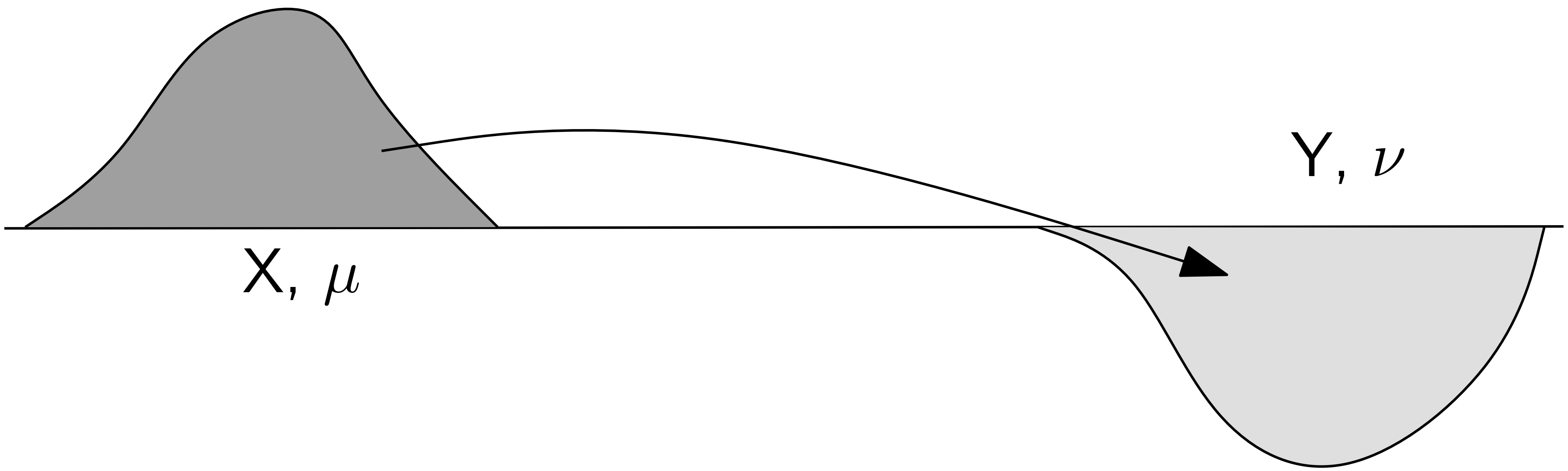

Let $c : X \times Y \to \mathbb{R}$ be a cost function e.g. $c(x, y) = \|x - y\|^2$

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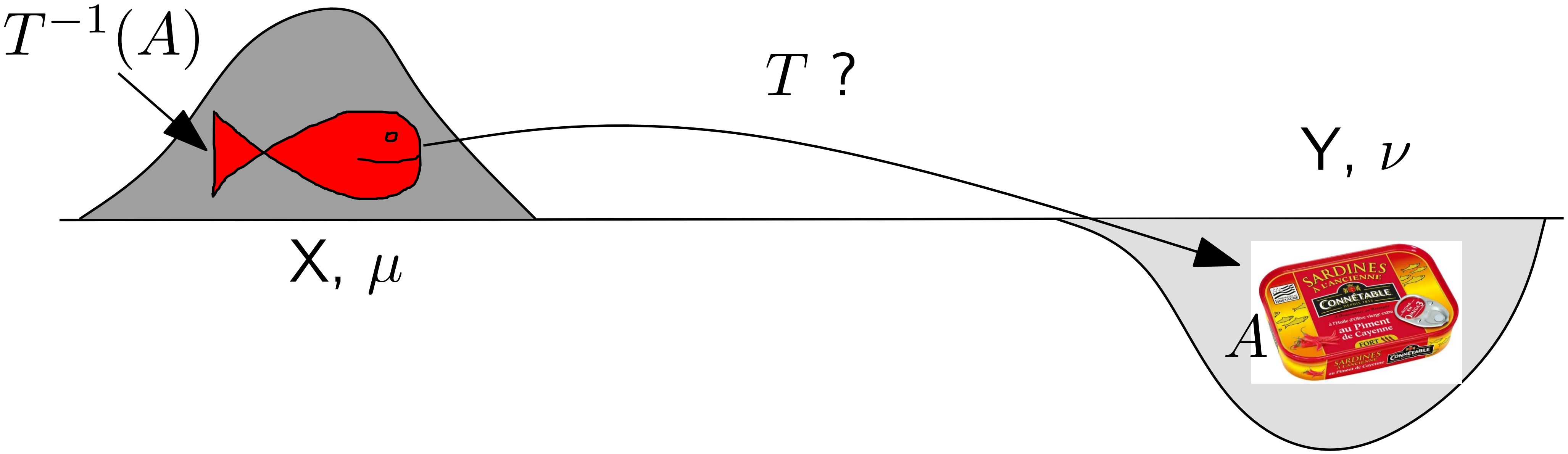

Let $c : X \times Y \to \mathbb{R}$ be a cost function   e.g. $c(x, y) = \|x - y\|^2$

Monge problem. Find a map $T : X \to Y$ such that

- T preserves the mass, i.e. $\nu(A) = \mu(T^{-1}(A))$
- T minimizes the total cost

$$\min \int_X c(x, T(x)) d\mu(x)$$

The minimizer does not always exist; Constraint not linear

Monge problem (1781)

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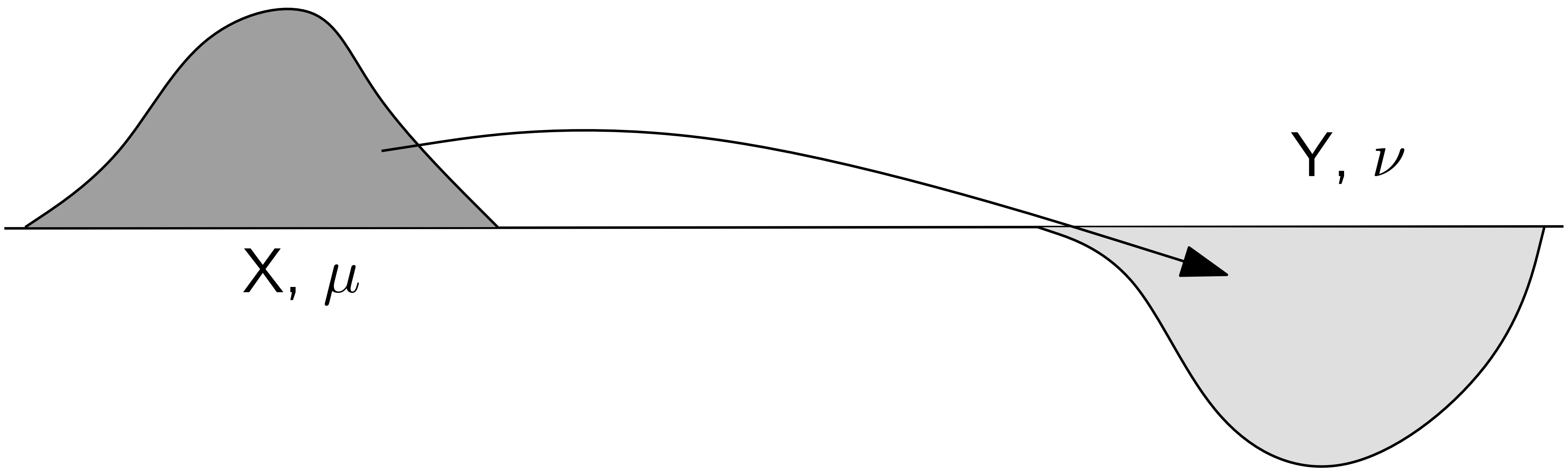


Let $c : X \times Y \to \mathbb{R}$ be a cost function e.g. $c(x, y) = \|x - y\|^2$

Kantorovitch relaxation – 1940's

Minimise $\int c(x, y) d\pi(x, y)$

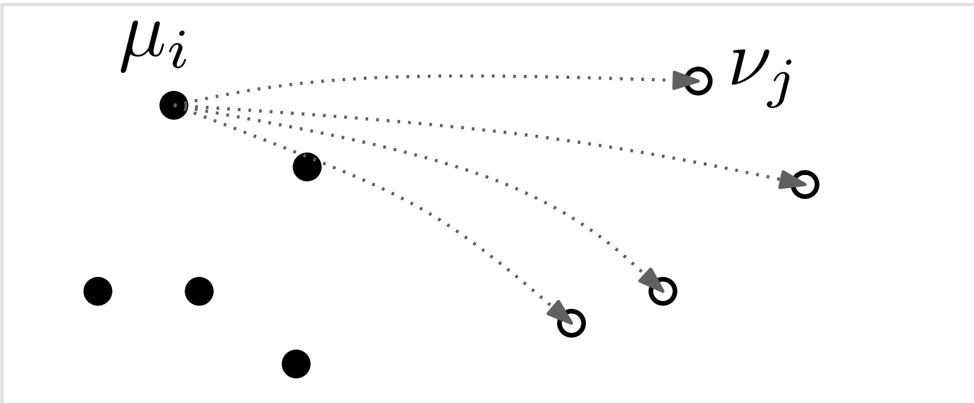
where π is a transport plan, i.e

π is a probability measure on $X \times Y$

$\pi(A \times Y) = \mu(A)$

$\pi(X \times B) = \nu(B)$

Numerical optimal transport



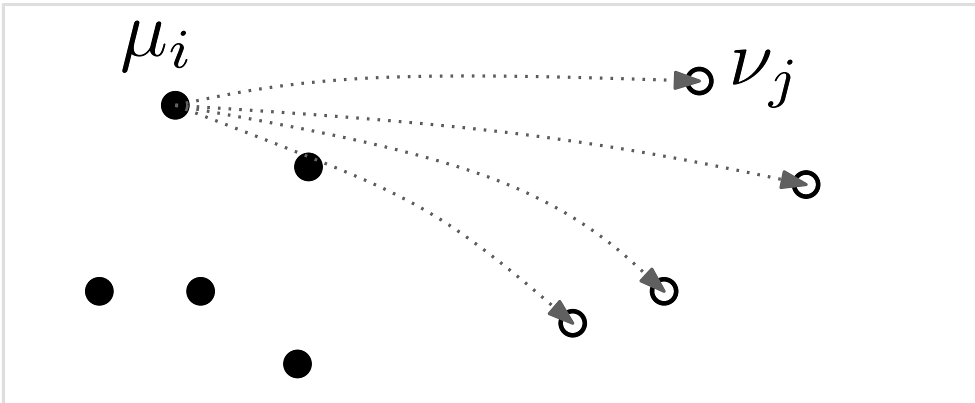
Discrete source and target

linear programming

Bertsekas' auction algorithm

Sinkhorn/IPFP

Numerical optimal transport



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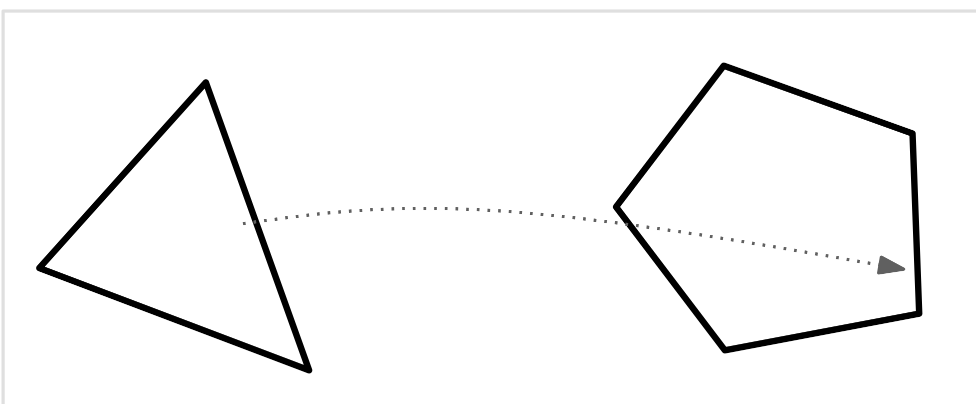
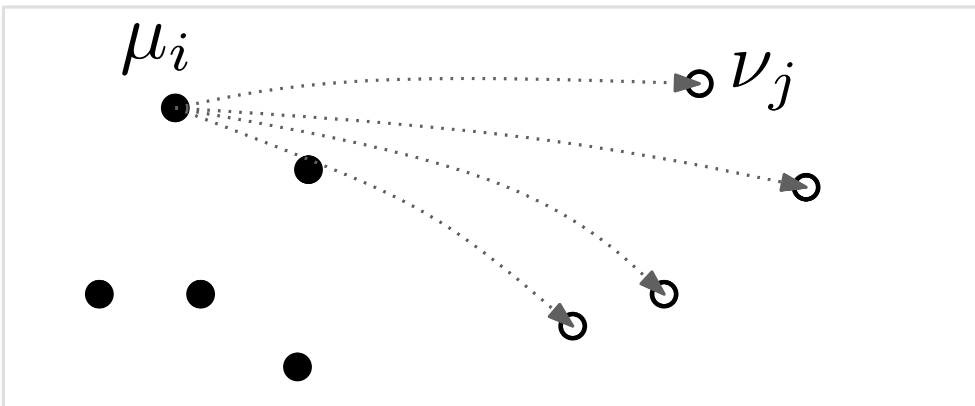


Source and target with density (PDE):

Benamou-Brenier formulation

Stencil methods for Monge Ampère equations

Numerical optimal transport



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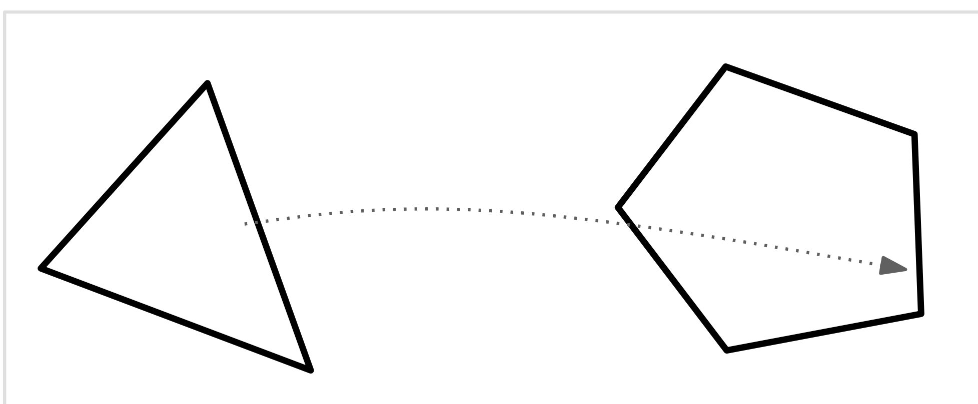

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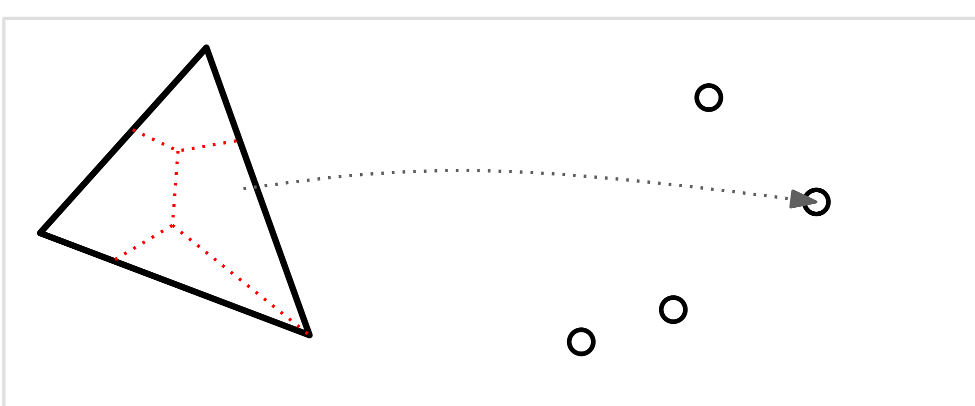

Source with density, discrete target:

Coordinate-wise increment
Oliker-Prussner '89 Caffarelli-Kochengin-Oliker '97
Kitagawa '12

Newton and quasi-Newton methods
Aurenhammer, Hoffmann, Aronov '98
Mérigot '11, Levy'15, Kitagawa-Mérigot-T.'17, etc.

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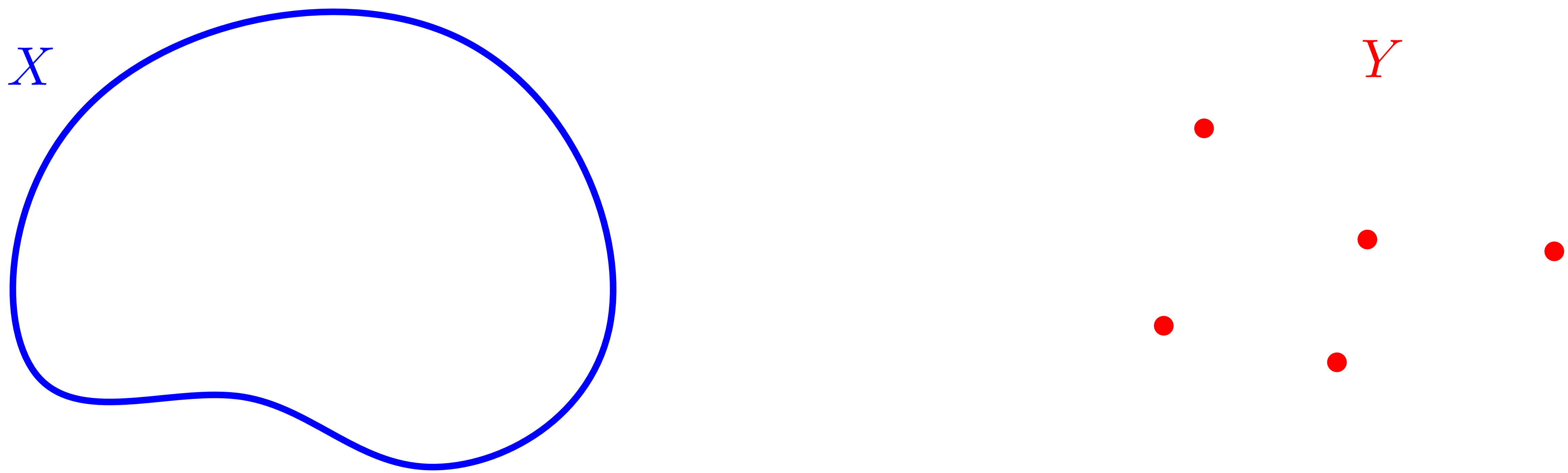
Semi-discrete optimal transport

$\mu(x) = \rho(x)\mathrm{d}x$ probability measure on X

$\nu = \sum_i \nu_i \delta_{y_i}$ prob. measure on finite $Y = \{y_1, \cdots, y_N\}$

$c : X \times Y \to \mathbb{R}$ cost function



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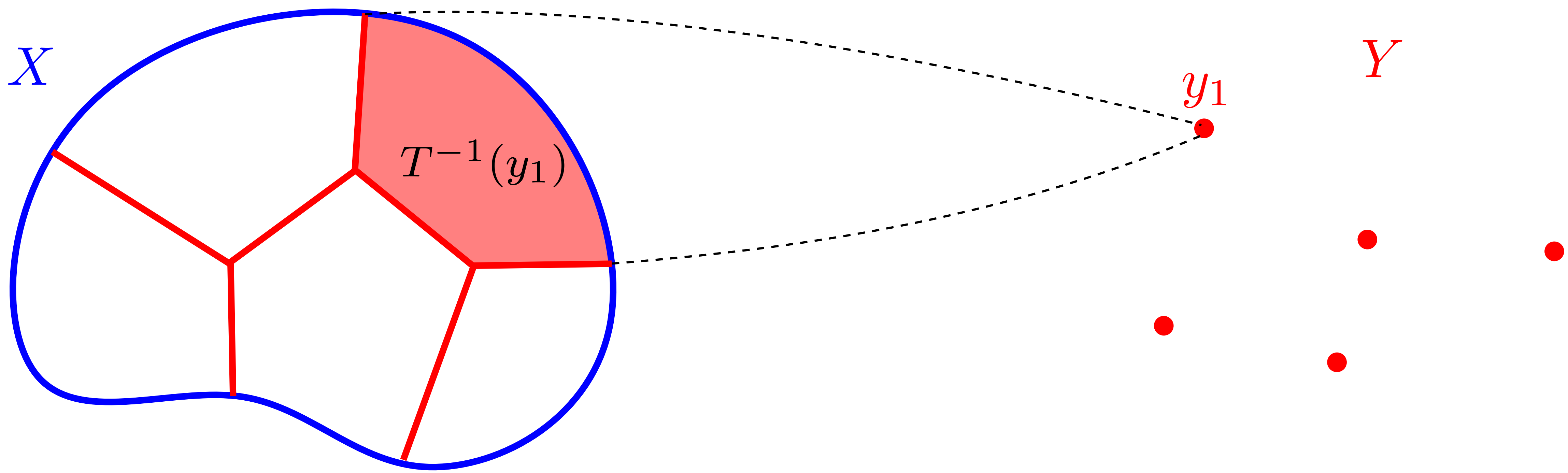

Transport map: $T : X \to Y$ s.t. $\forall i,\ \mu(T^{-1}(\{y_i\})) = \nu_i$ (i.e. $T_\# \mu = \nu$)

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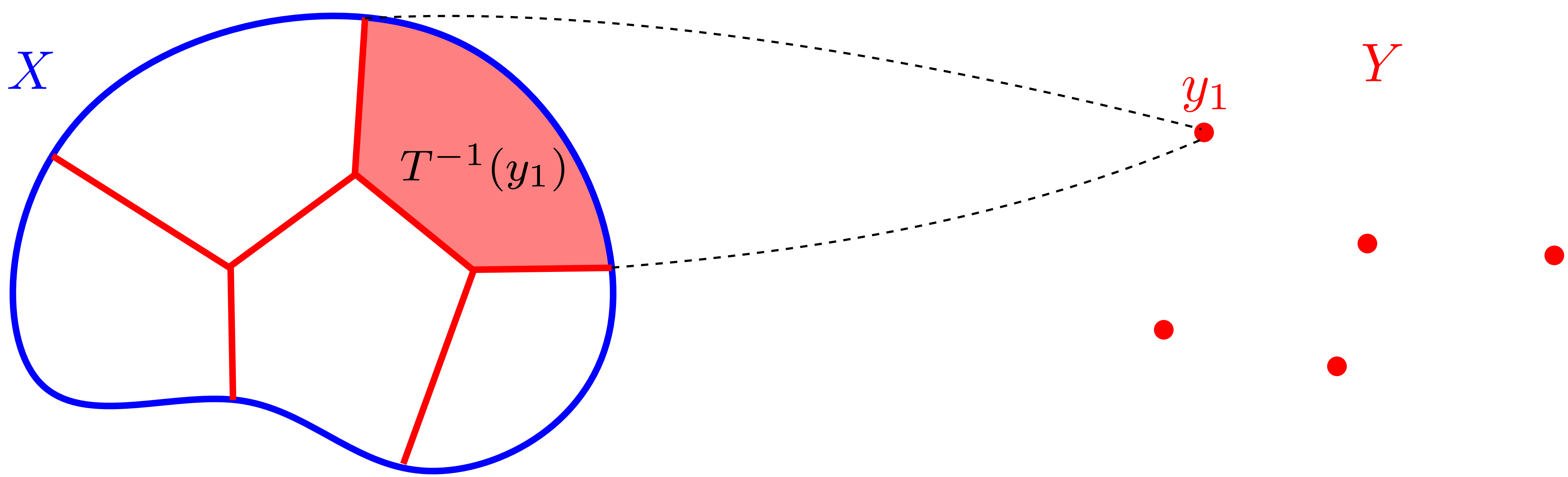

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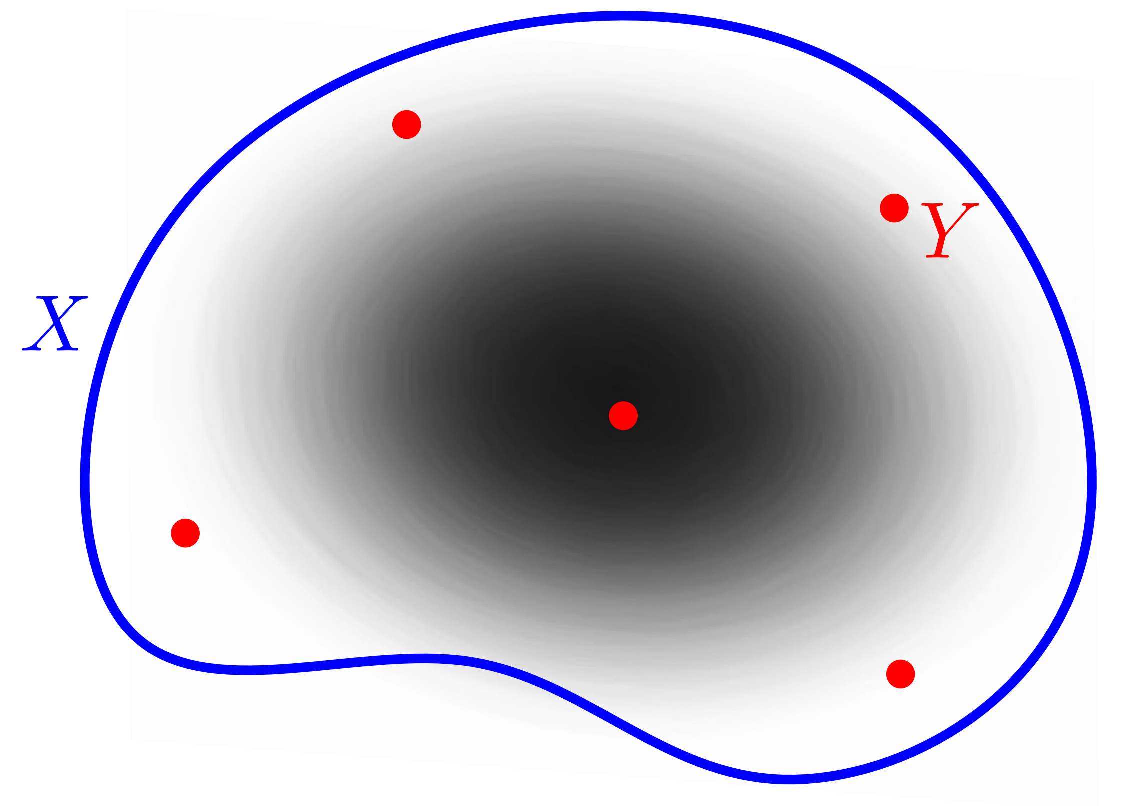
Semi-discrete optimal transport

$\rho : X \to \mathbb{R}$ density of population

Y = location of bakeries

$c(x, y_i) := \|x - y_i\|^2$



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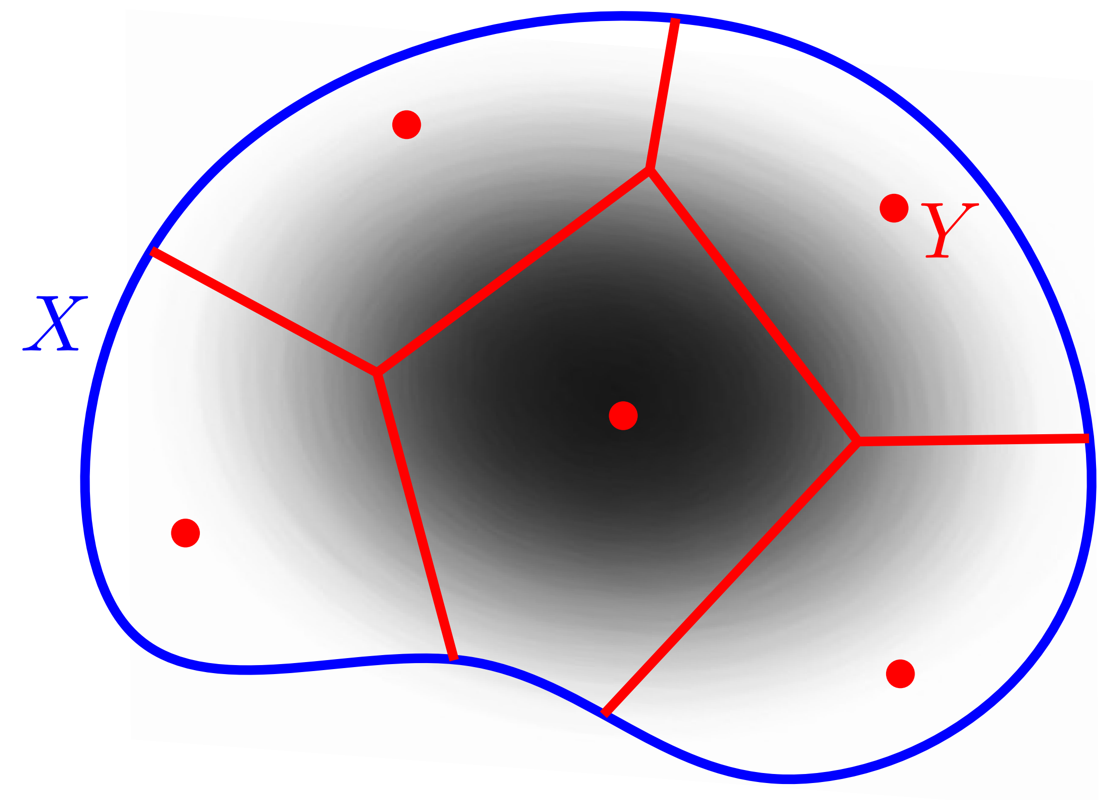

- If the price of bread is uniform, people go the closest bakery:

$$\boxed{\mathrm{Vor}(y_i) = \{x \in X; \forall j,\ c(x, y_i) \leq c(x, y_j)\}}$$

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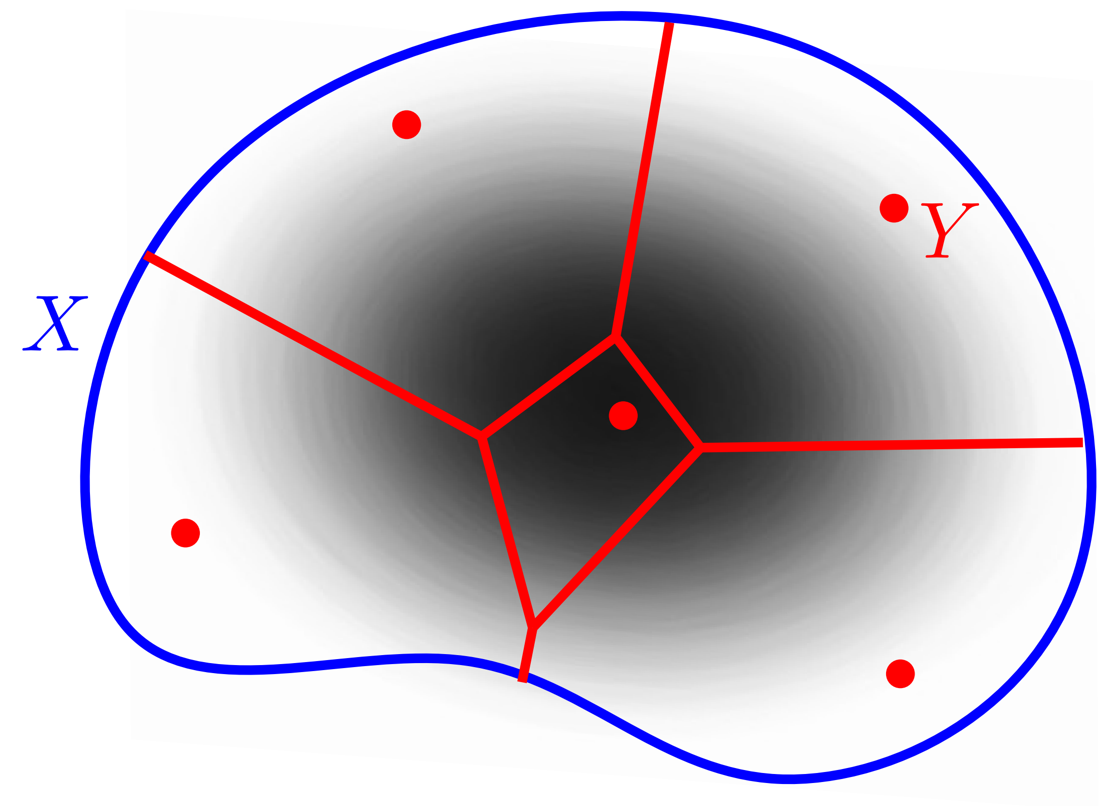

- If prices are given by $\psi_1, \cdots, \psi_N$, people make a compromise:

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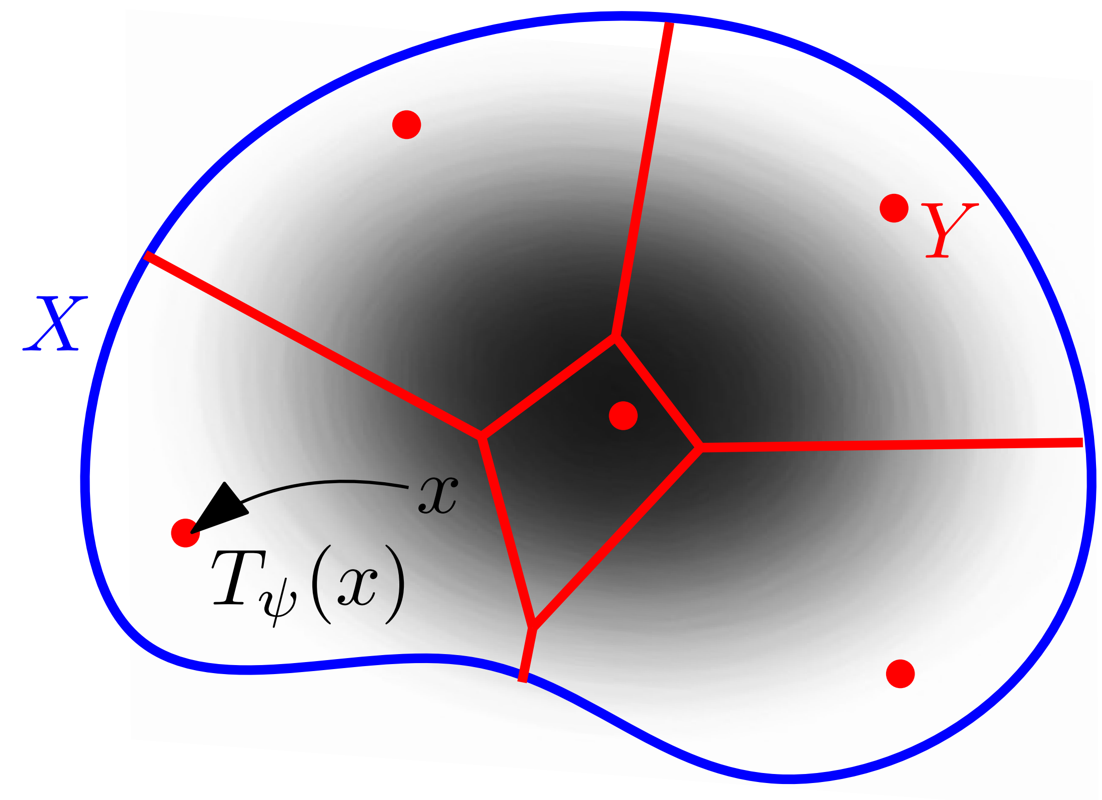

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Lemma: The map $T_\psi : X \to Y$ is an **optimal transport map** between ρ and ν_ψ where $\nu_{\psi,i} = \rho(\mathrm{Lag}_i(\psi))$ is the measure of $\mathrm{Lag}_i(\psi)$

Semi-discrete optimal transport

$\rho : X \to \mathbb{R}$ density of population

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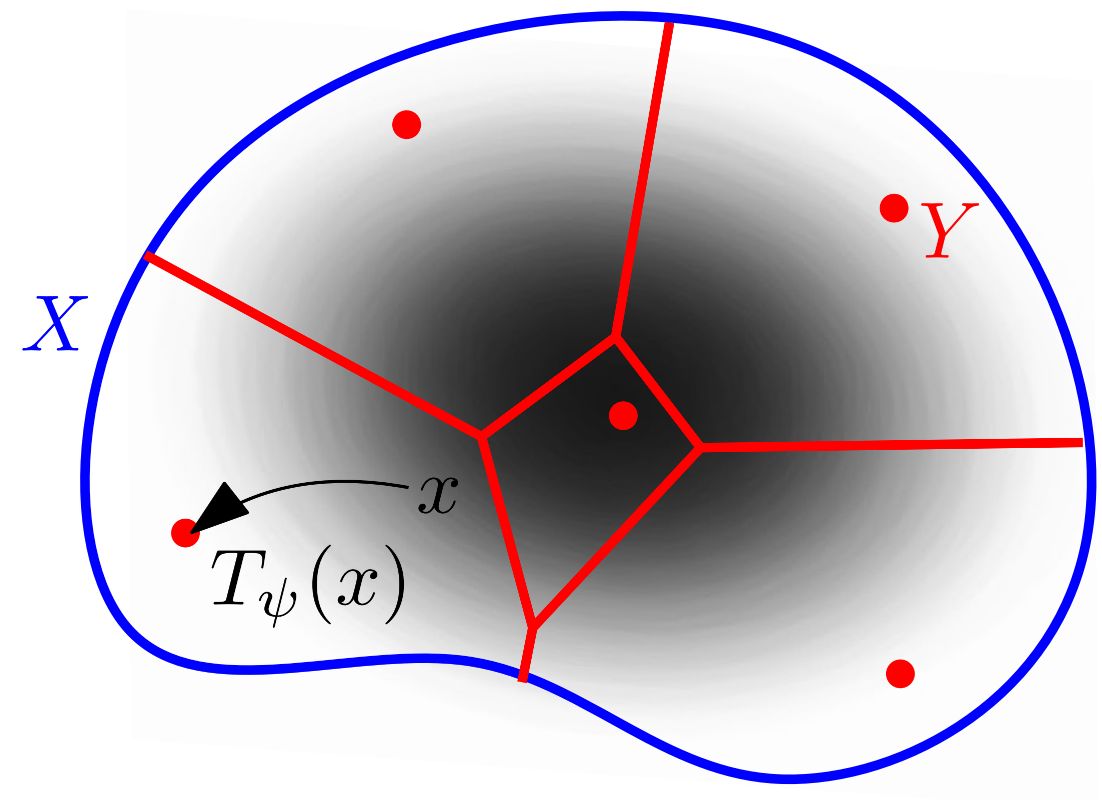

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For other costs c, **(Twist)**: $\forall x$, the map $y \mapsto \nabla_x c(x, y)$ is injective

Semi-discrete optimal transport

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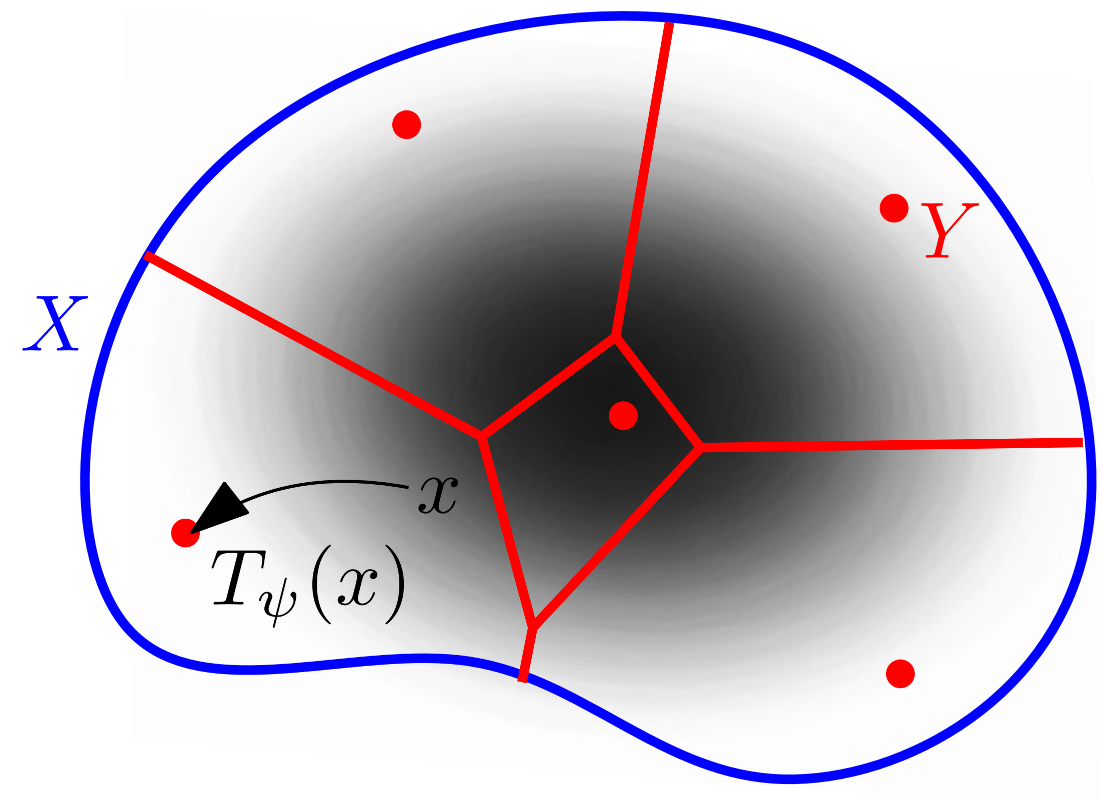

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For other costs c, **(Twist)**: $\forall x$, the map $y \mapsto \nabla_x c(x, y)$ is injective

Solving OT between ρ and $\nu \iff$ Finding ψ s.t. $\rho(\mathrm{Lag}_i(\psi)) = \nu_i\ \forall i$

Kantorovitch duality

Theorem: Finding an **optimal transport** between ρ and $\nu = \sum_i \nu_i \delta_{y_i}$

$\iff$ maximizing the **concave** function $\Phi : \mathbb{R}^N \to \mathbb{R}$

$$\Phi(\psi) := \sum_i \int_{\mathrm{Lag}_i(\psi)} [c(x, y_i) + \psi_i] \,\mathrm{d}\,\rho(x) - \sum_i \psi_i \nu_i$$

Aurenhammer, Hoffman, Aronov '98

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- Quasi Newton methods for $c(x,y) = \|x-y\|^2$ on $\mathbb{R}^2/\mathbb{R}^3$ $\mathbb{S}^2$ No analysis

  [Mérigot. '11] [Lévy '14] [de Goes et al '12] [Machado, Mérigot, Thibert '16]
- Newton method in $\mathbb{R}^2$, $\mathbb{R}^3$, when μ supported on a triangulation.

Outline

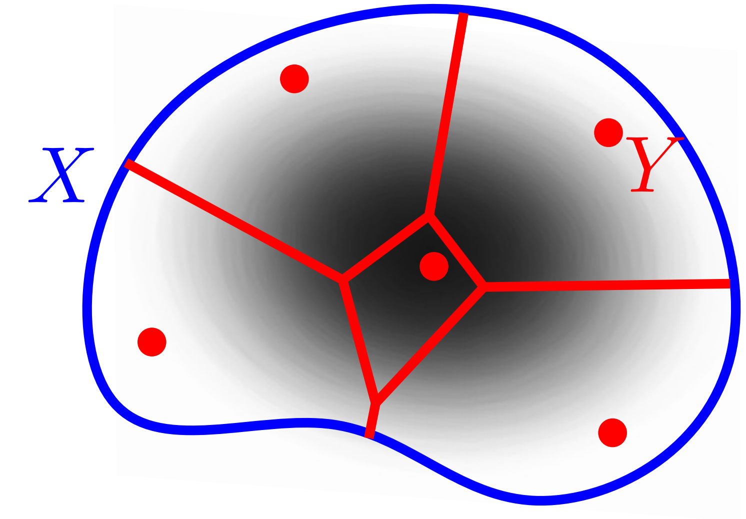
- Case 1: mirror for point light source
- Case 2: mirror for collimated source light

- Optimal transport
- Semi-discrete optimal transport
- Damped Newton algorithm

- Non-imaging optics: Far-Field target
- Non-imaging optics: Near-Field target

Newton Algorithm

Equation $\rho(\mathrm{Lag}_i(\psi)) = \nu_i$ for all i



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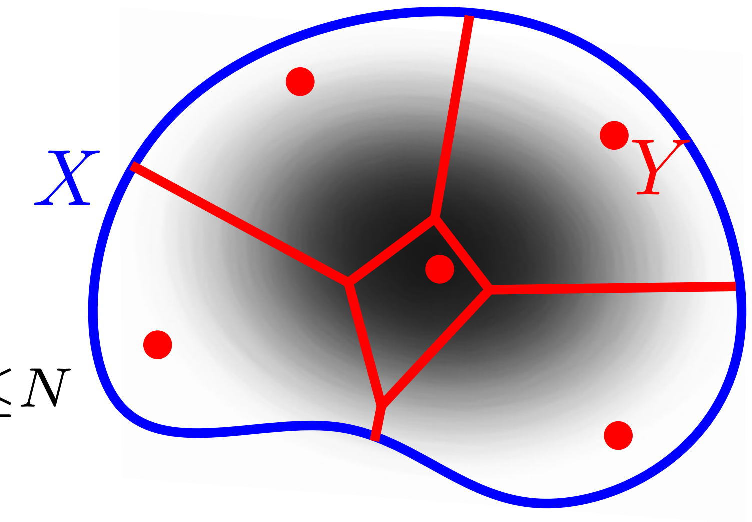

We define $G : \mathbb{R}^N \to \mathbb{R}^N$ by $G(\psi) = (\rho(\mathrm{Lag}_i(\psi)))_{1\leq i \leq N}$

Equation $G(\psi) = \nu$

Remark: G is invariant by addition of a vector $\lambda(1, \cdots, 1)$.

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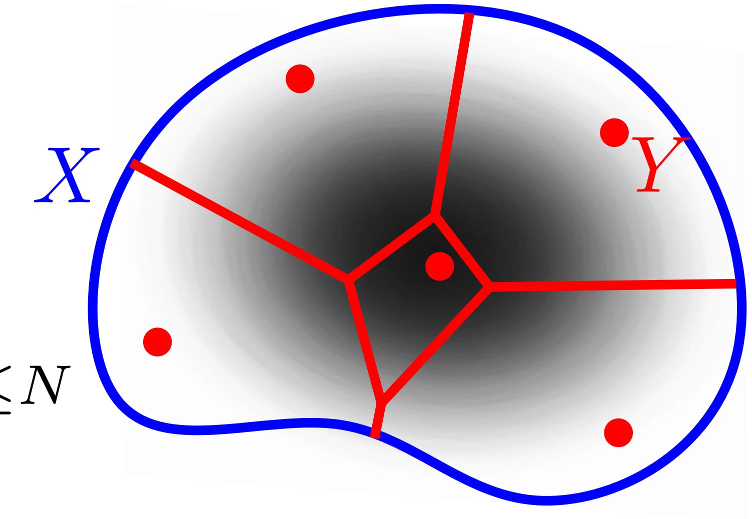

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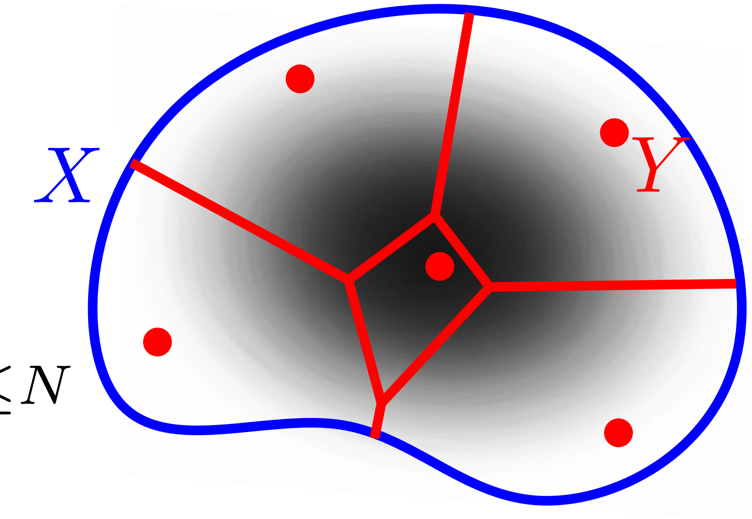

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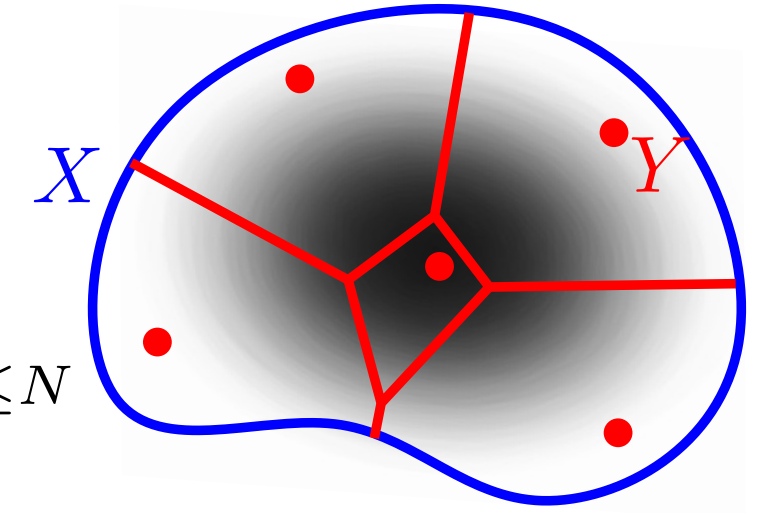

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How about global convergence ?

Remark: If $\mathrm{Lag}_i(\psi) = \emptyset$ then $DG_i(\psi) = 0$ locally and d^k not unique.

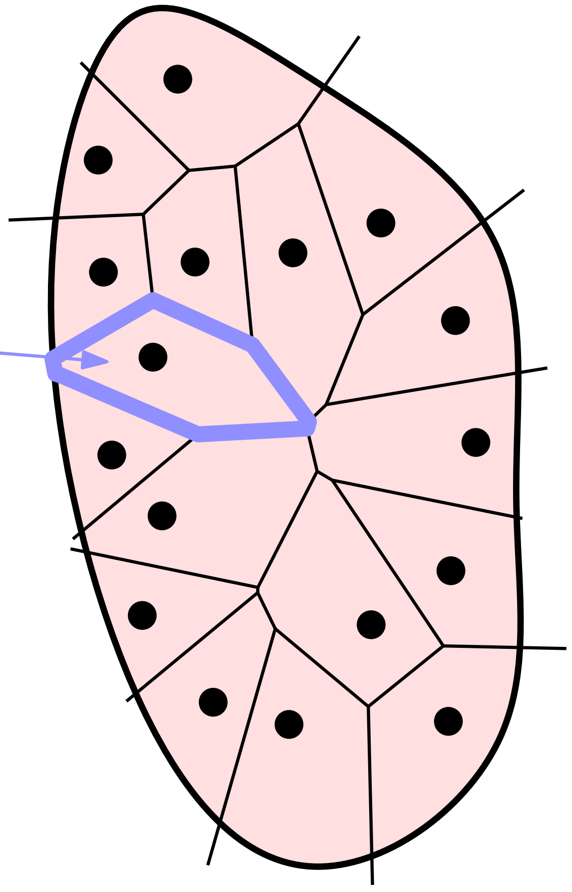
We want to enforce $\mathrm{Lag}_i(\psi^k) \neq \emptyset$.

Damped Newton Algorithm

Equation $G(\psi) = \nu$ where $G(\psi) = (\rho(\mathrm{Lag}_i(\psi)))_{1 \leq i \leq N}$

Admissible domain: $E_\varepsilon := \{\psi \in \mathbb{R}^N ; \forall i, \rho(\mathrm{Lag}_i(\psi) \geq \varepsilon\}$

$\rho(\mathrm{Lag}_i(\psi)) \geq \varepsilon$



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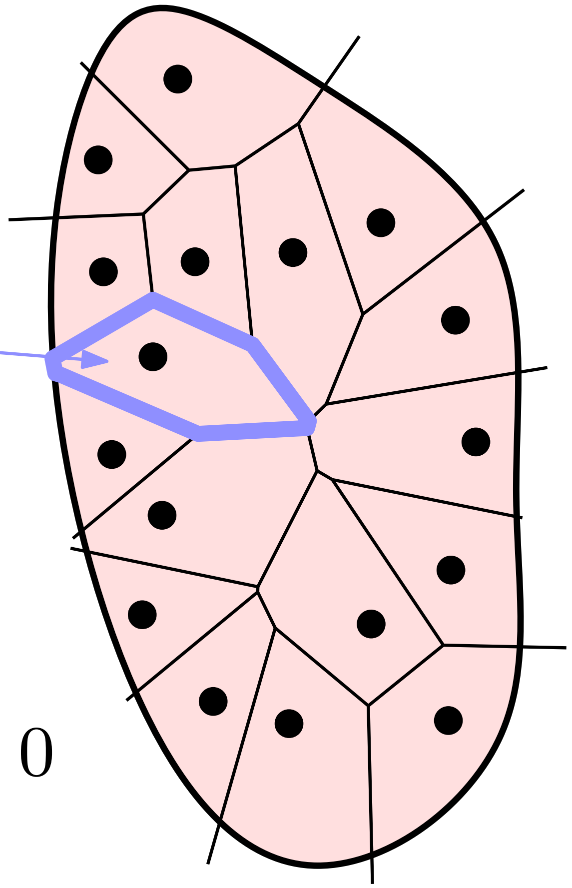

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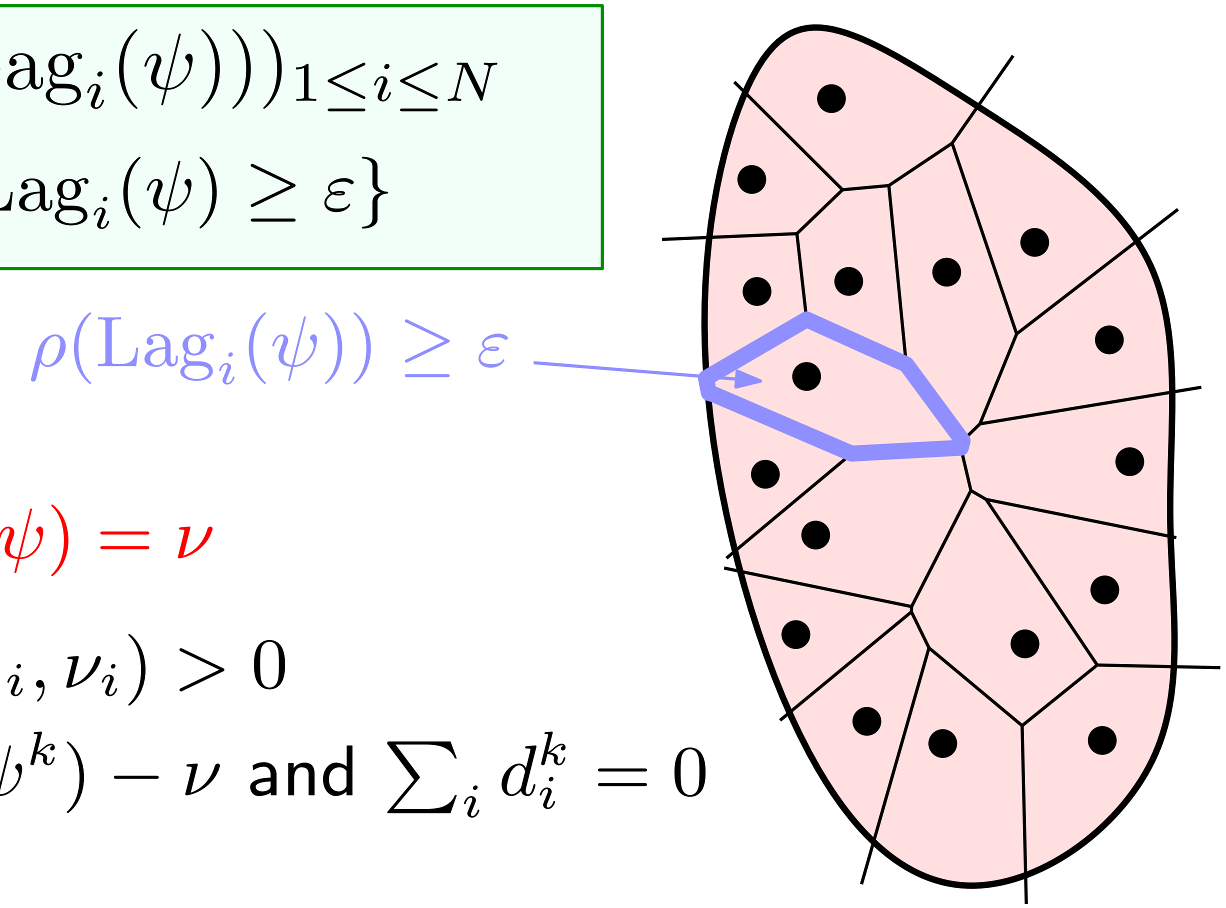

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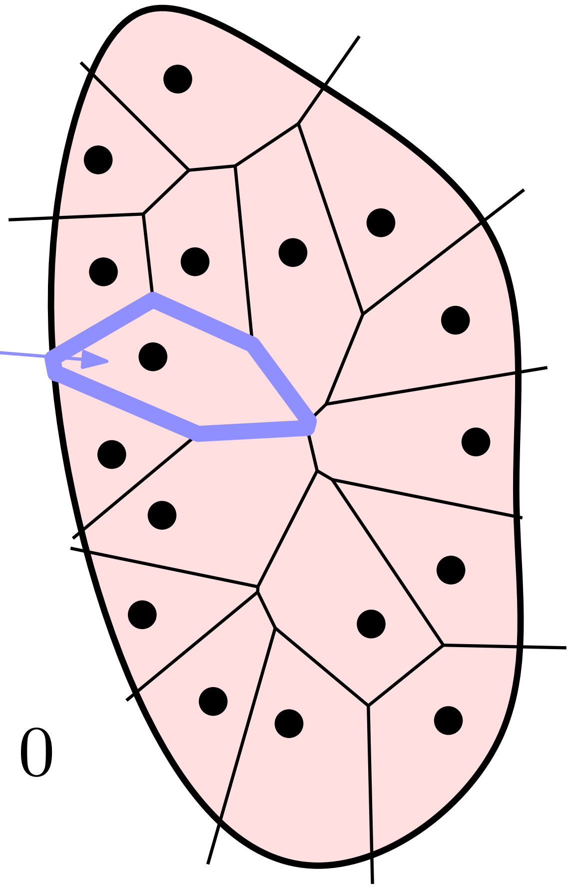

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cf [Mirebeau '15]

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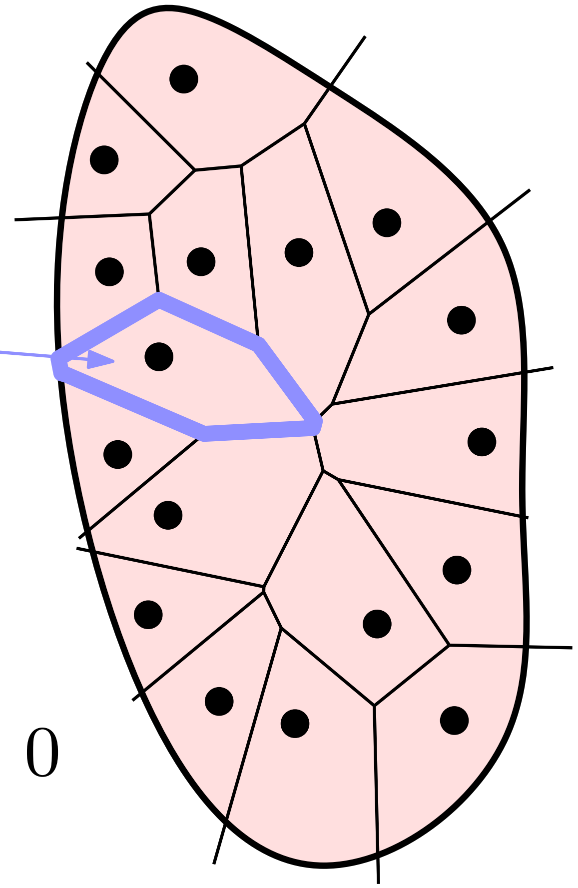

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$\Rightarrow$ We have to show smoothness and strict monotonicity

Quadratic cost : smoothness of G

we have $G_i(\psi) = \rho(\mathrm{Lag}_i(\psi))$ $\quad c(x,y) := \|x-y\|^2$

Proposition: For $\psi \in E_\varepsilon$, and assuming that $\rho \in \mathcal{C}^0_c(\mathbb{R}^d)$ one has

(A) $\frac{\partial G_i}{\partial \psi_j}(\psi) = \frac{1}{2\|y_i - y_j\|} \int_{\boxed{\mathrm{Lag}_{ij}(\psi)}} \rho(x)\,\mathrm{d}\,x$ (B) $\quad \frac{\partial G_i}{\partial \psi_i}(\psi) = -\sum_{j\neq i} \frac{\partial G_i}{\partial \psi_j}(\psi)$

$j \neq i$

$\mathrm{Lag}_{ij}(\psi) := \mathrm{Lag}_i(\psi) \cap \mathrm{Lag}_j(\psi)$

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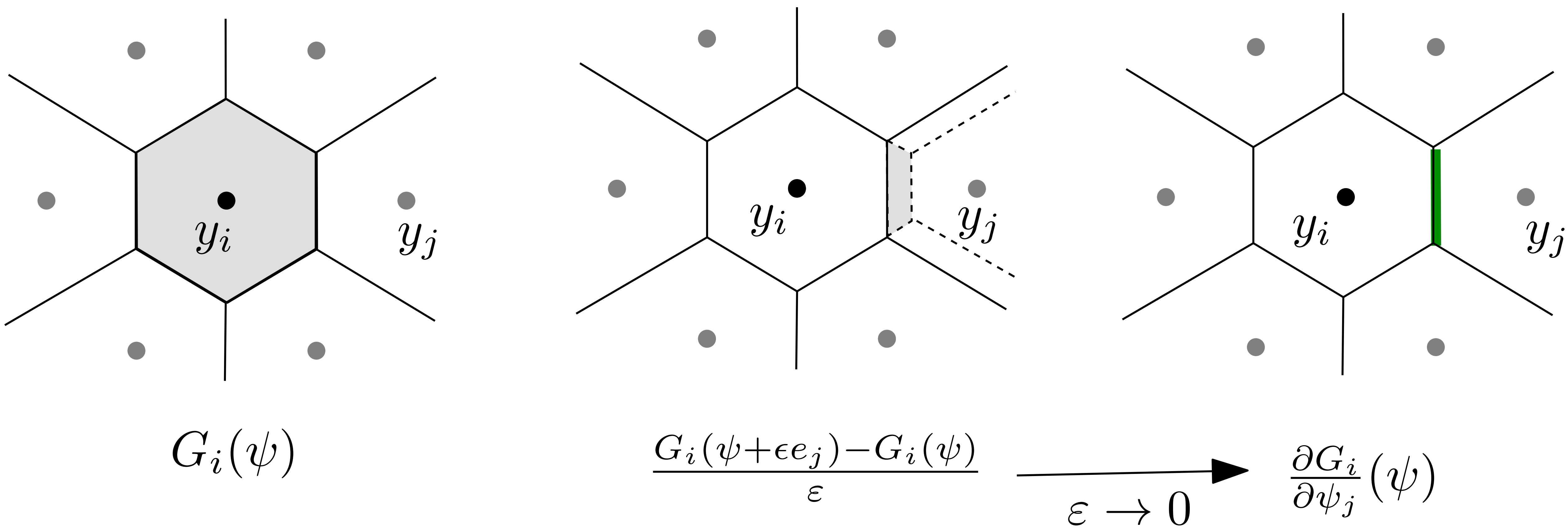
Intuition of the proof:



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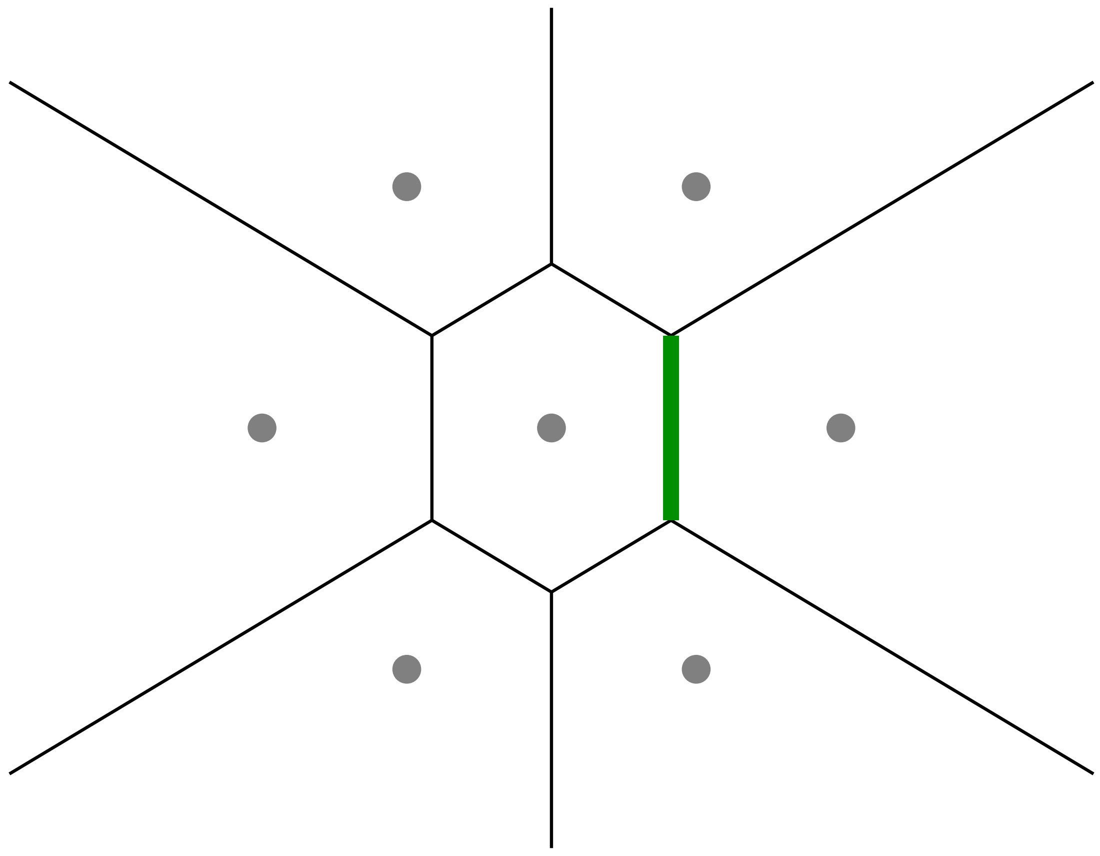

Continuity of $\frac{\partial G_i}{\partial \psi_j}(\psi)$

When t varies, $\frac{\partial G_i}{\partial \psi_j}(\psi_t)$ increases ...

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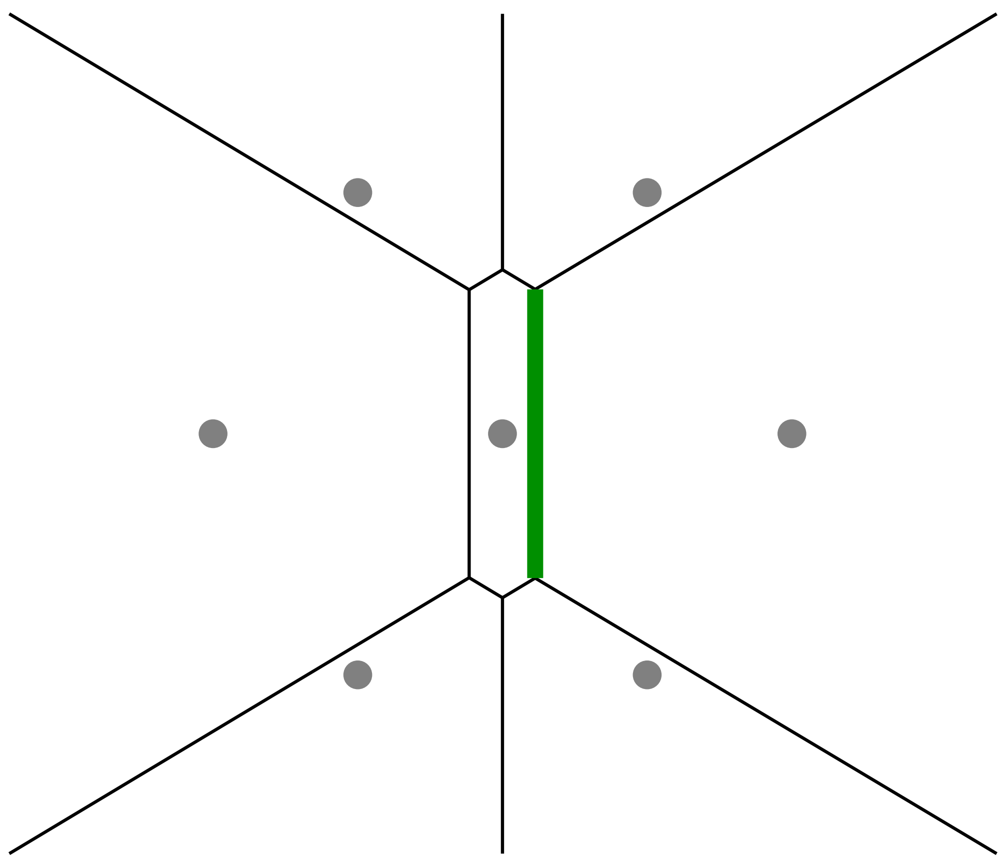

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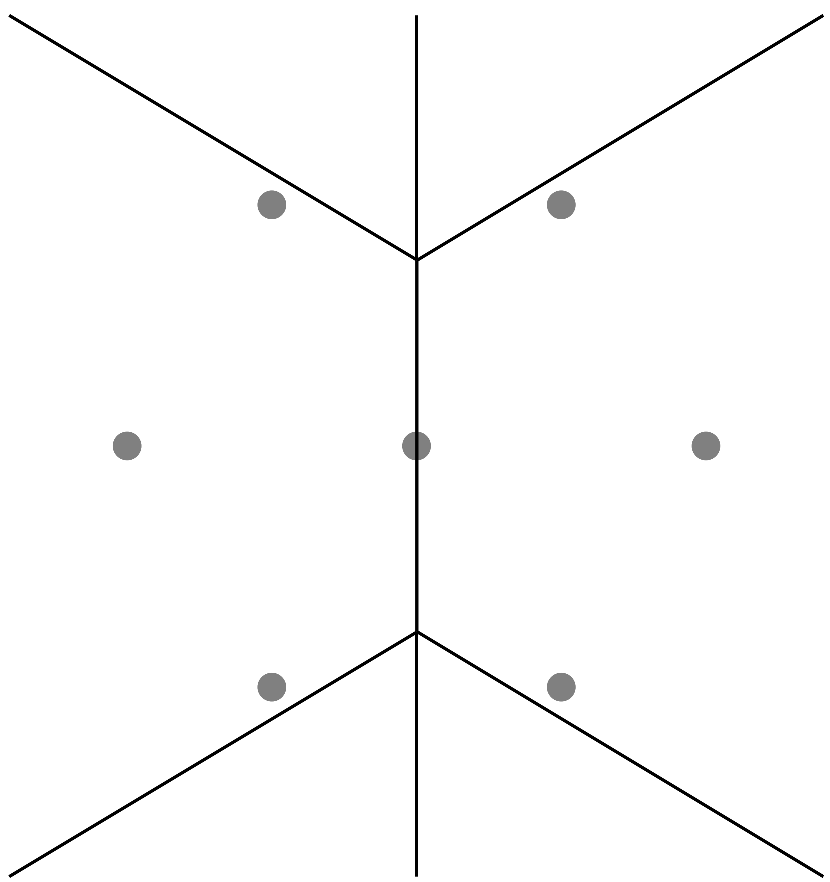

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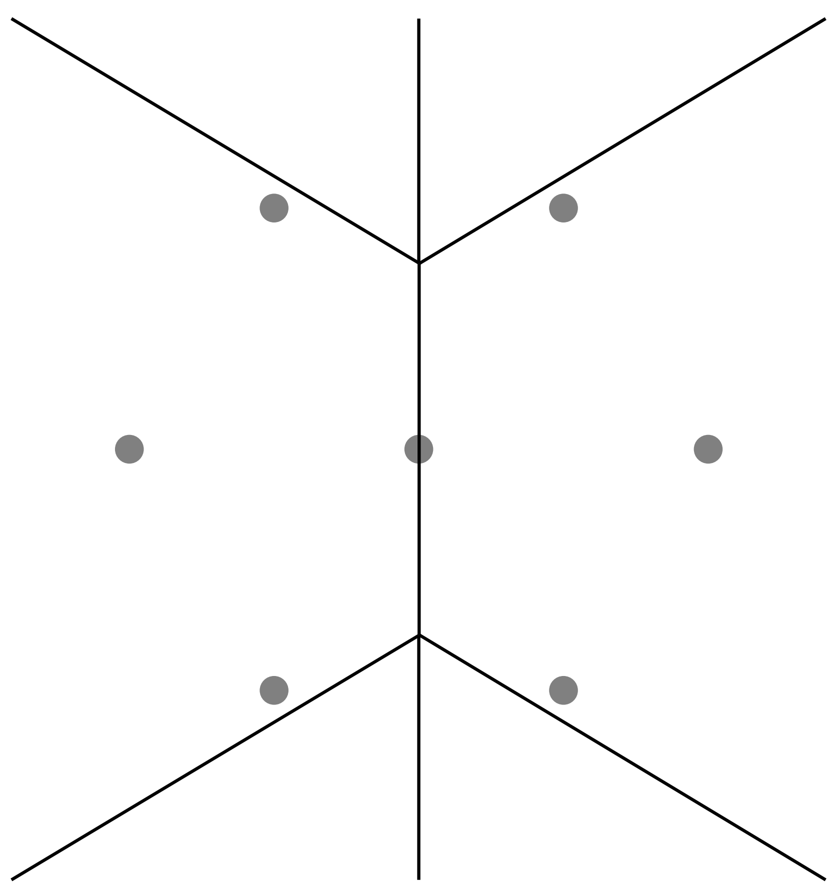

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$\rightsquigarrow$ we require $-$ $\rho(\mathrm{Lag}_i(\psi)) > 0$ at all times

$-$ or a genericity condition (three points not aligned)

Quadratic cost: strict monotonicity of G

we have $G_i(\psi) = \rho(\mathrm{Lag}_i(\psi))$

Recall: $\frac{\partial G_i}{\partial \psi_j}(\psi) = \int_{\boxed{\mathrm{Lag}_{ij}(\psi)}} \frac{\rho(x)\,\mathrm{d}\,x}{2\|y_i - y_j\|}$ $\qquad$ $\frac{\partial G_i}{\partial \psi_i}(\psi) = -\sum_{j \neq i} \frac{\partial G_i}{\partial \psi_j}(\psi)$

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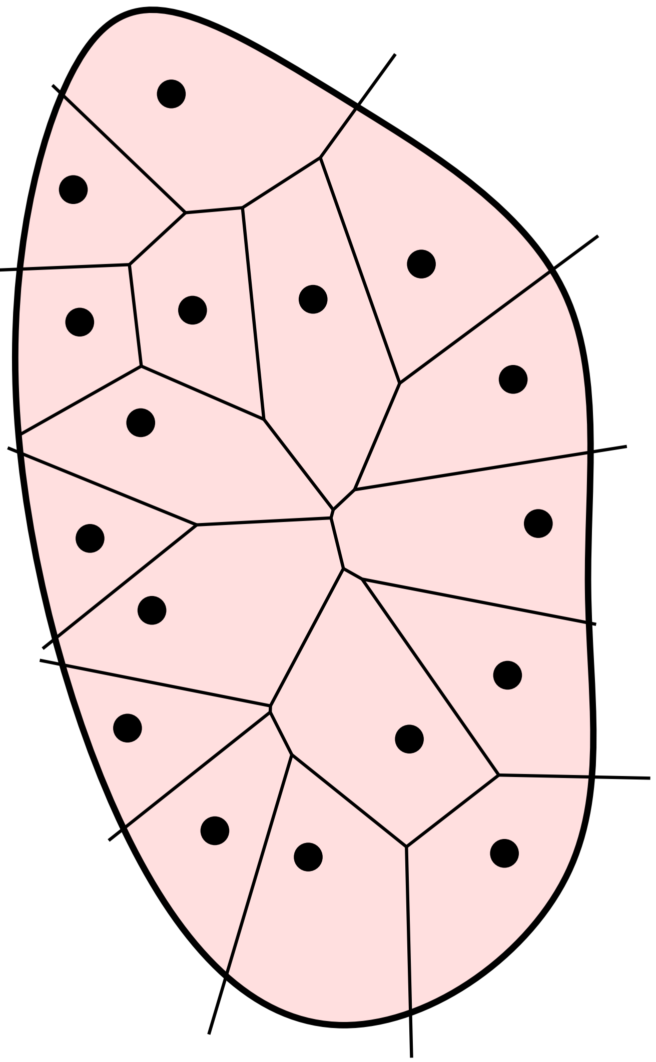

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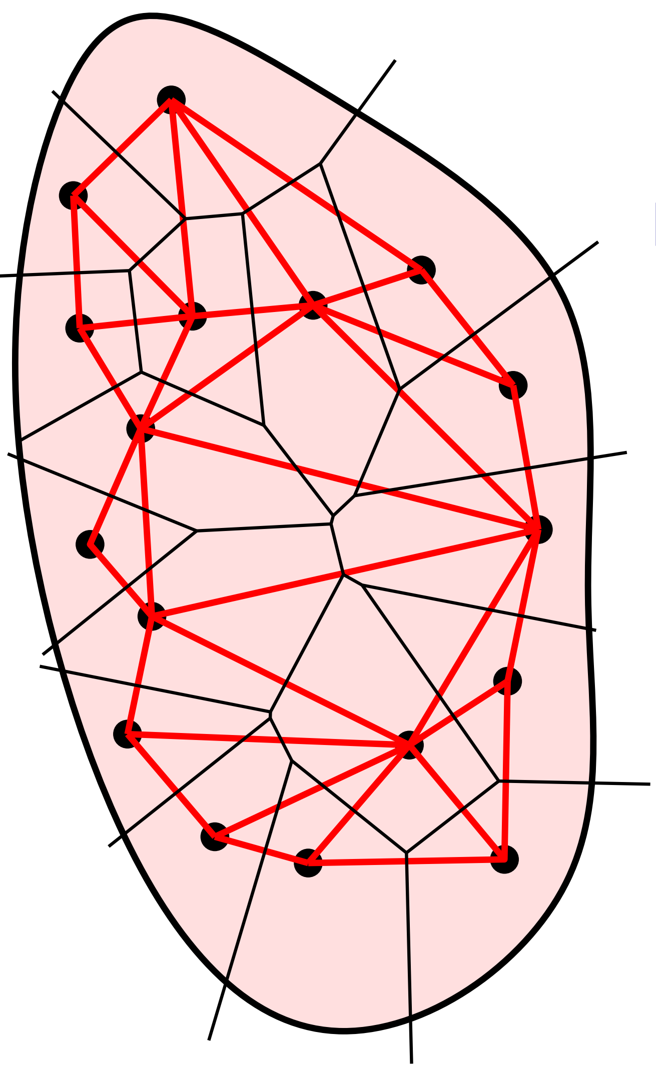

- Consider the matrix $(L_{ij}) := \frac{\partial G_i}{\partial \psi_j}(\psi)$ and the graph H:

$$\boxed{(y_i, y_j) \in H \iff L_{ij} > 0 \iff \mathrm{Lag}_{ij}(\psi) \cap \{\rho > 0\} \neq \emptyset.}$$

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Recall: $\frac{\partial G_i}{\partial \psi_j}(\psi) = \int_{\boxed{\mathrm{Lag}_{ij}(\psi)}} \frac{\rho(x)\,\mathrm{d}\,x}{2\|y_i - y_j\|}$ $\qquad$ $\frac{\partial G_i}{\partial \psi_i}(\psi) = -\sum_{j \neq i} \frac{\partial G_i}{\partial \psi_j}(\psi)$

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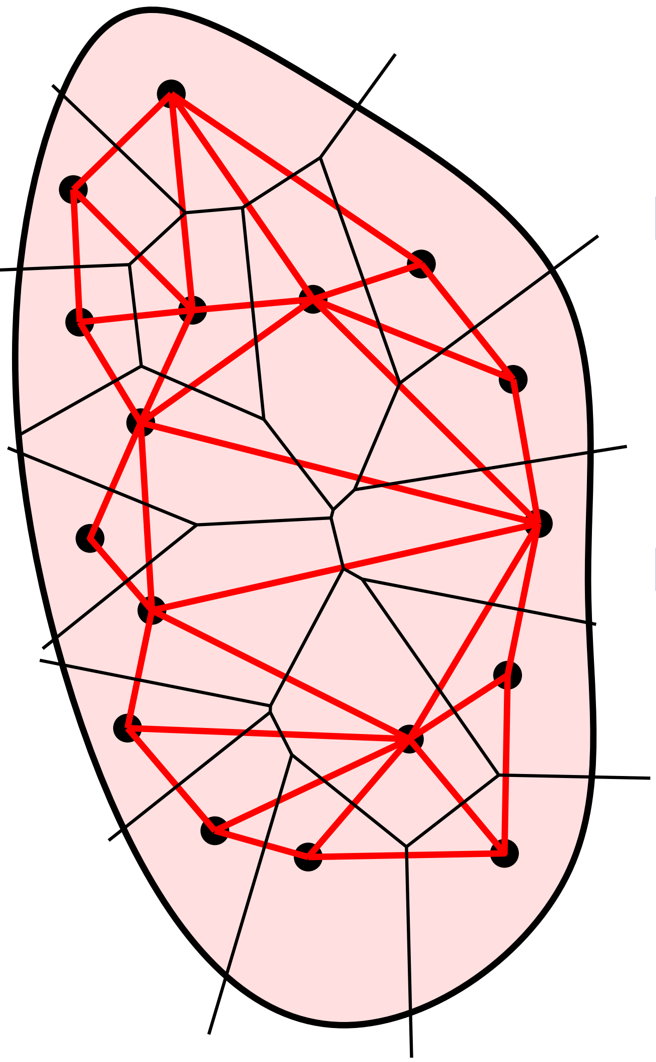

- Consider the matrix $(L_{ij}) := \frac{\partial G_i}{\partial \psi_j}(\psi)$ and the graph H:

$$\boxed{(y_i, y_j) \in H \iff L_{ij} > 0 \iff \mathrm{Lag}_{ij}(\psi) \cap \{\rho > 0\} \neq \emptyset.}$$

- If $\{\rho > 0\}$ is connected and $\psi \in E_\varepsilon$, then H is connected.

Quadratic cost: strict monotonicity of G

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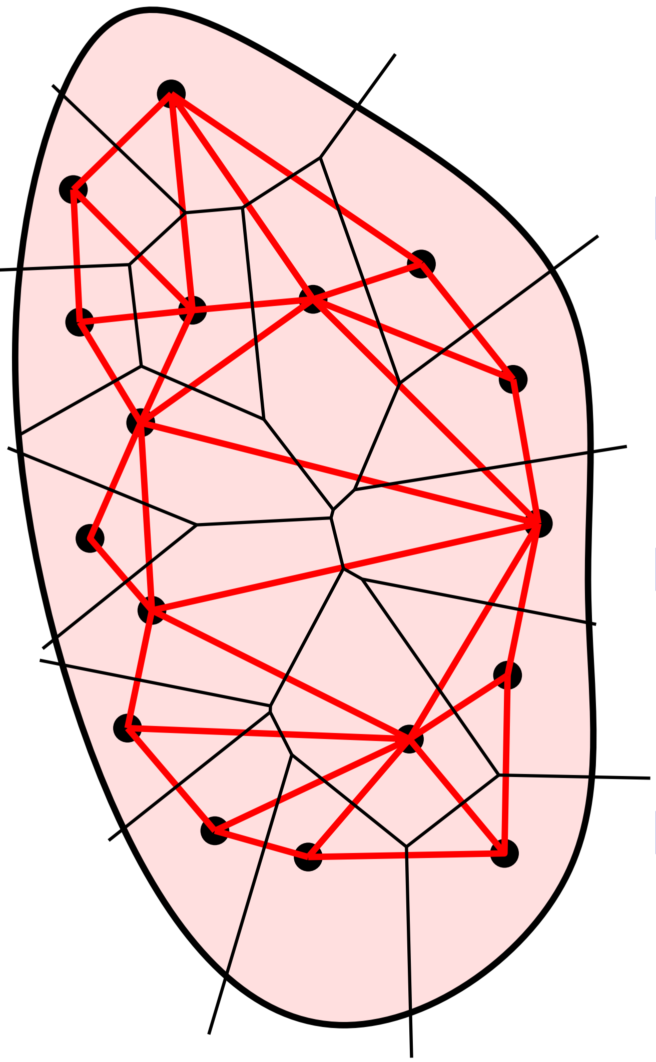

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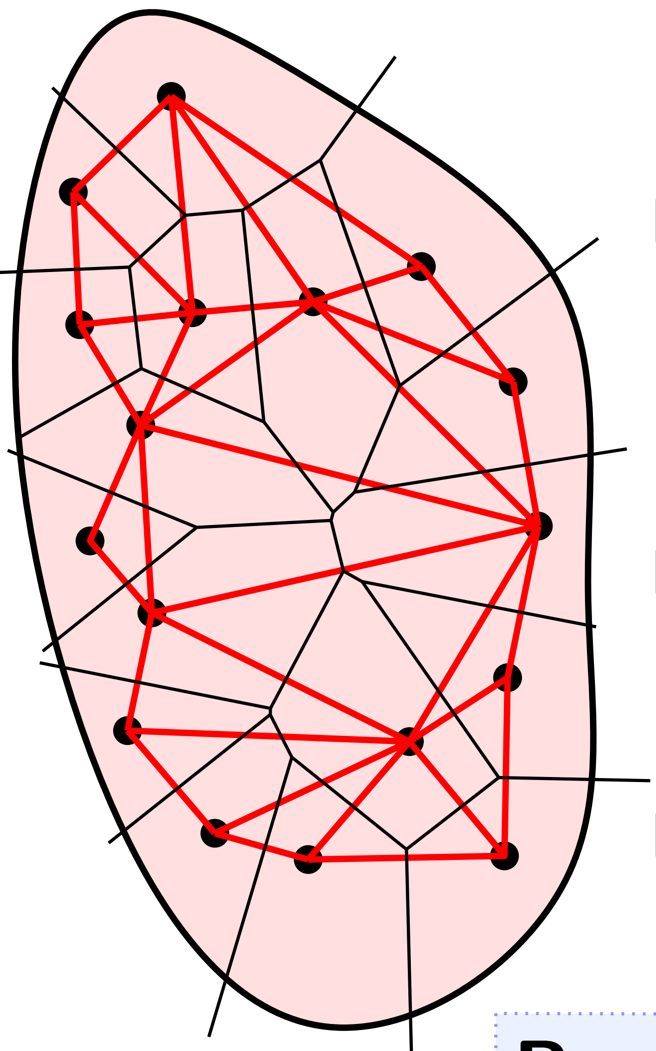

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- $\mathrm{Ker}(L) = \{cst\} = \mathbb{R}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Proposition: Assume $\rho \in \mathcal{C}^0_c(\mathbb{R}^d)$ and $\{\rho > 0\}$ connected. Then,

$\forall \psi \in E_\varepsilon, \ \ \forall v \in \{cst\}^\perp \ \ \langle DG(\psi)v | v\rangle < 0$

$\rightsquigarrow$ we require connectedness conditions on ρ

Convergence in the quadratic case

Theorem: Let X be a (closed) convex bounded domain of $\mathbb{R}^d$ with $Y \subset \mathbb{R}^d$ be a finite set, ρ of class C^1 and $\{\rho > 0\}$ connected.

Then, the damped Newton algorithm for SD-OT converges **globally** with linear rate and locally with quadratic rate.

$$\|G(\psi^{k+1}) - \nu\| \leq \left(1 - \frac{\tau^*}{2}\right)^2 \|G(\psi^k) - \nu\|$$

[Kitagawa, Mérigot, T., JEMS 2017]

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- Holds when $X \subset \mathbb{R}^d$, c satistifes Twist.
  No convexity assumption but genericity conditions [Mérigot, T., 2020]

Quadratic cost: numerics

Exemple: ρ uniform on $X = [0,1]^2$; $\nu = \frac{1}{N}\sum_i \delta_{y_i}$

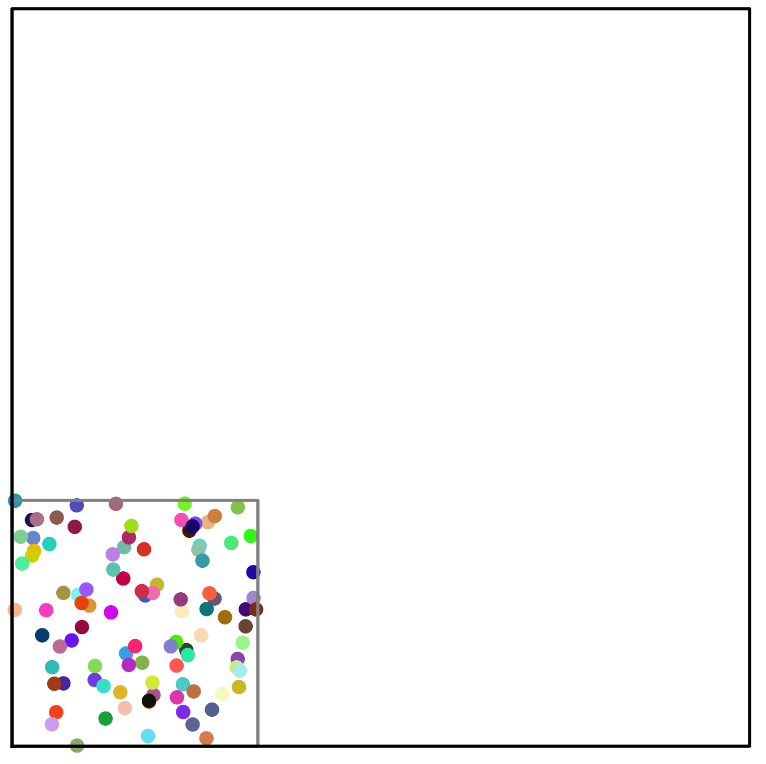

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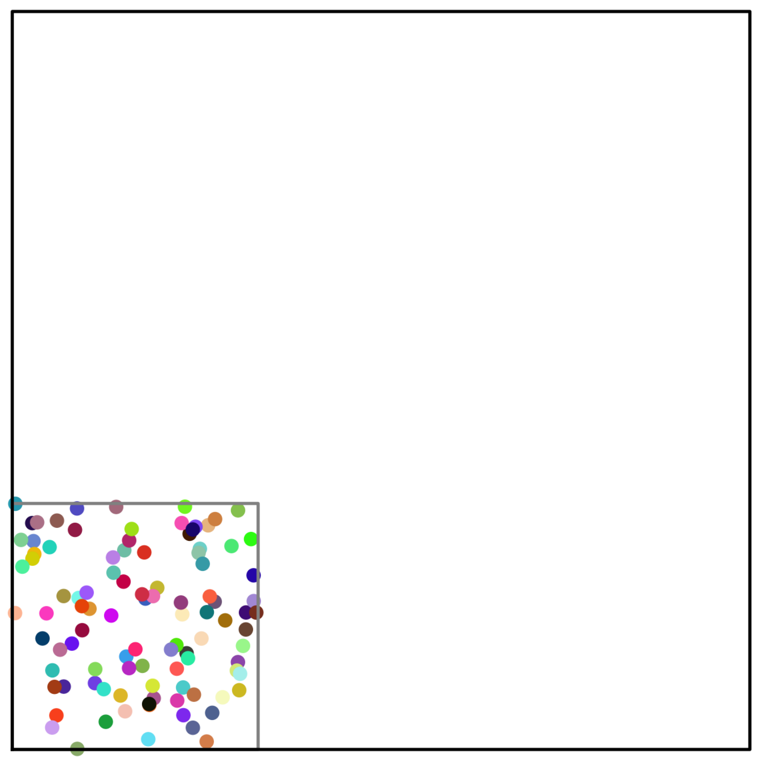
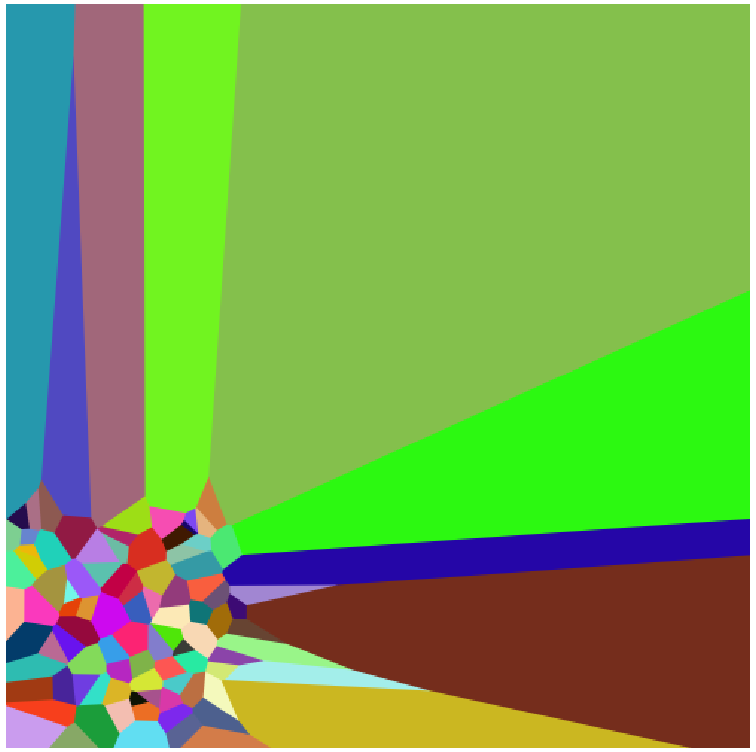
diagramme de Laguerre





$\|G(\psi^0) - \nu\|_1$
$\simeq 1.8$

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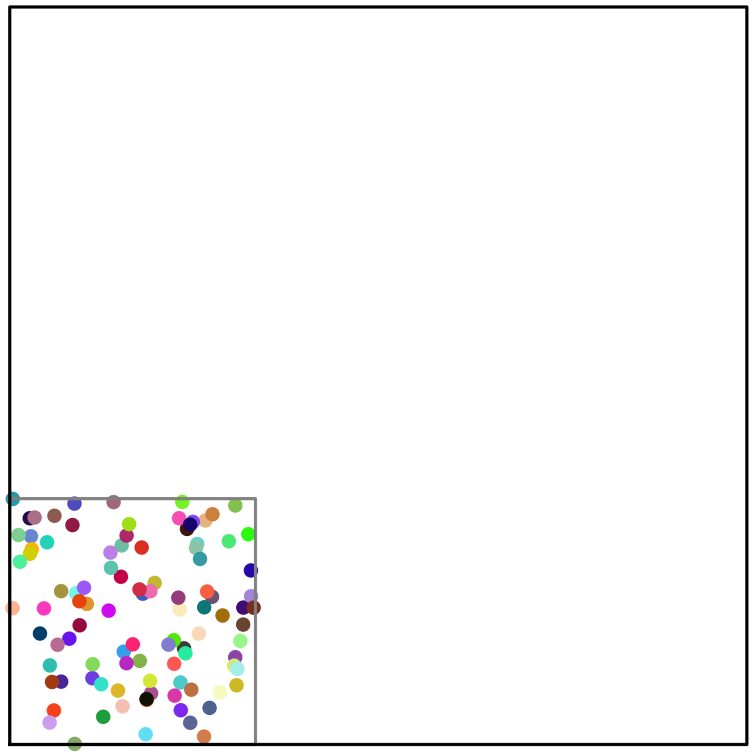
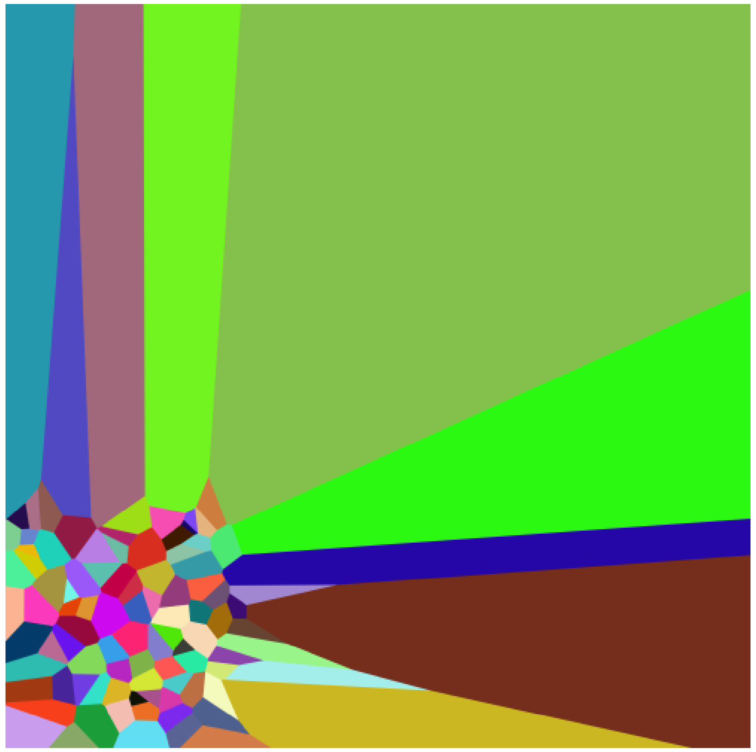
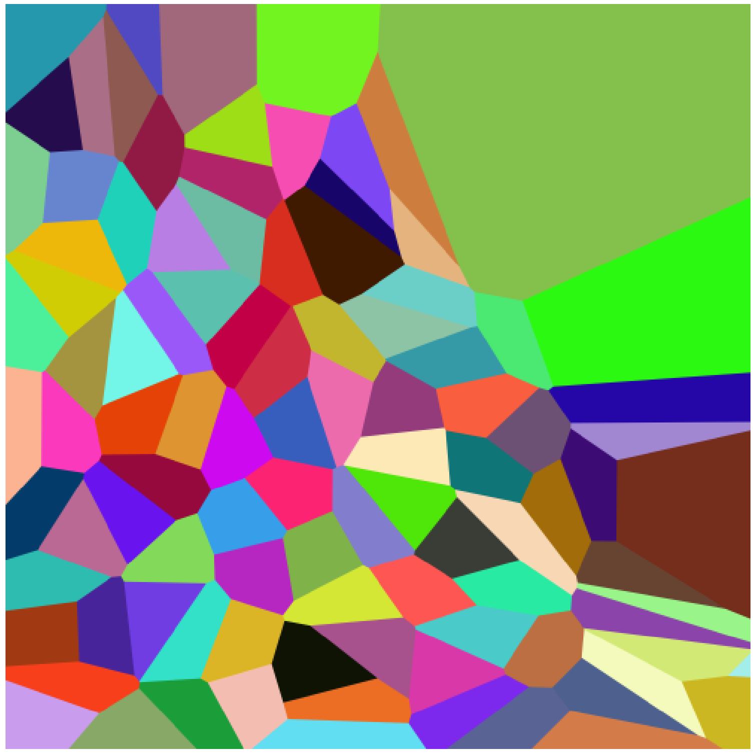




$\|G(\psi^0) - \nu\|_1$
$\simeq 1.8$

$\|G(\psi^1) - \nu\|_1$
$\simeq 0.6$



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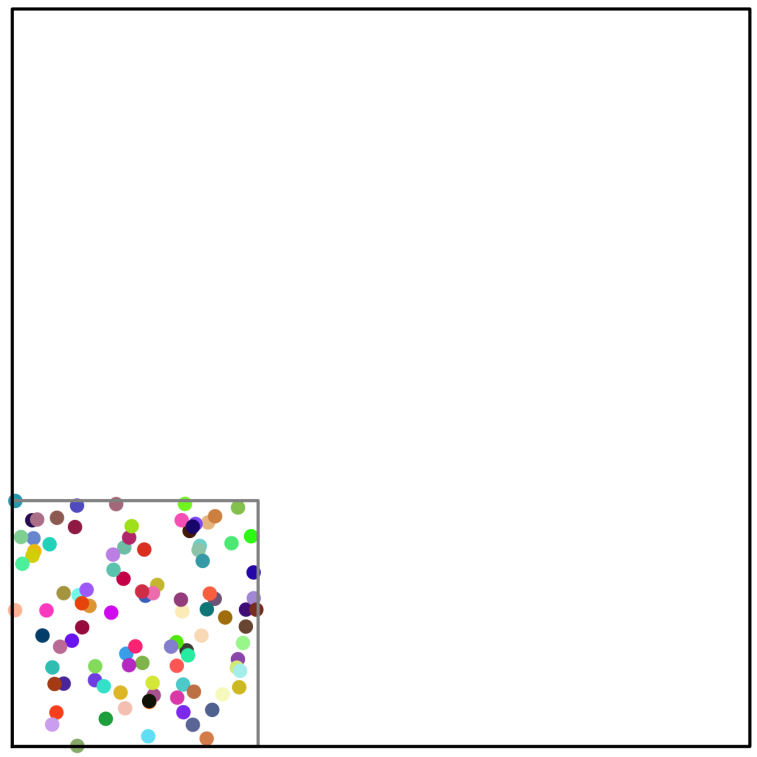

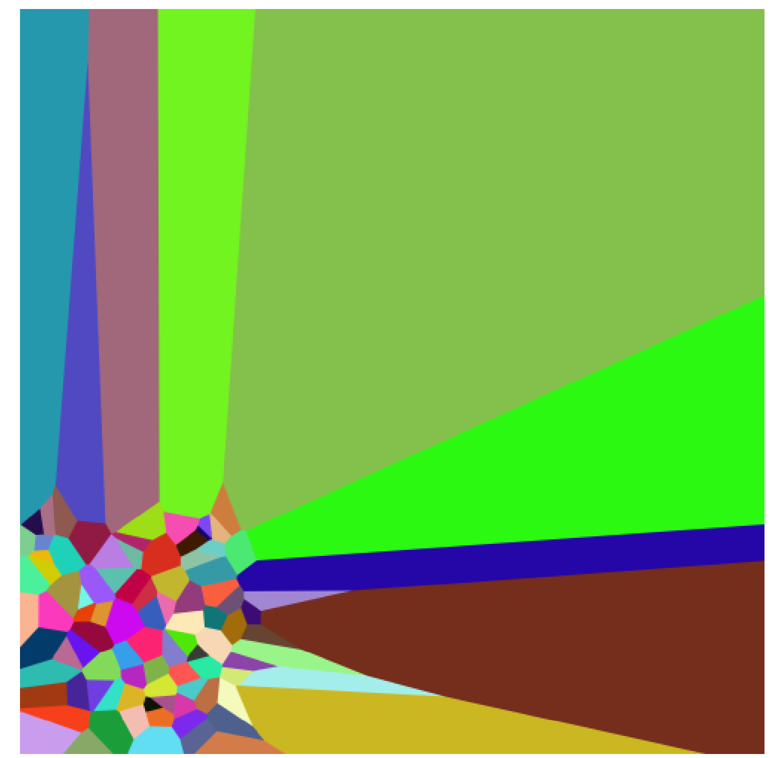

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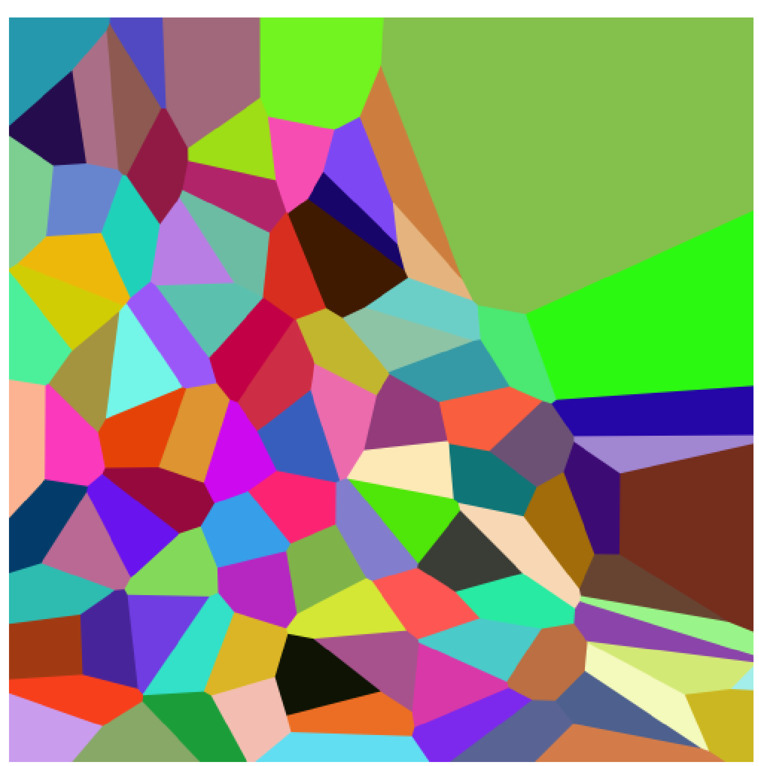
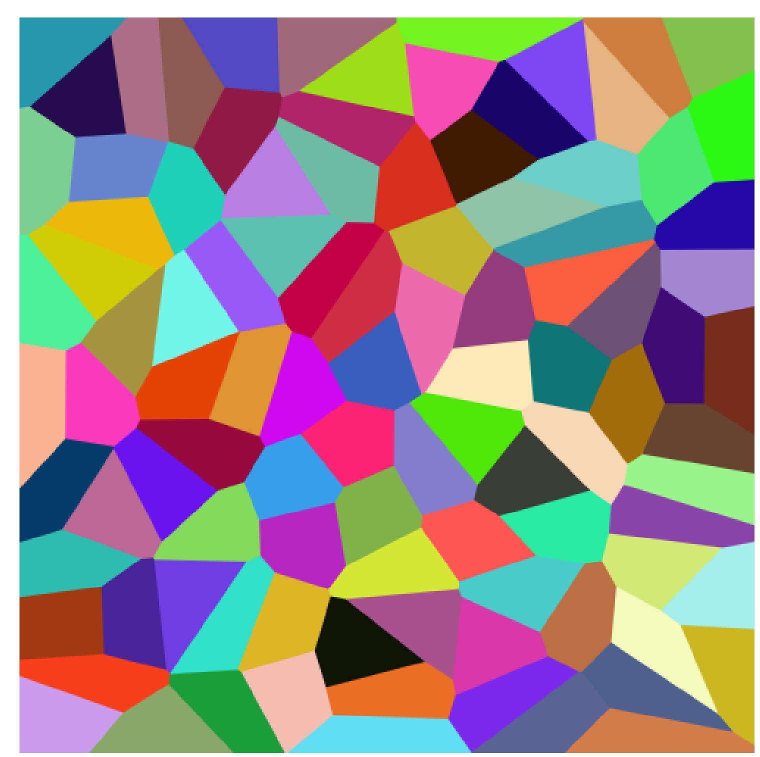
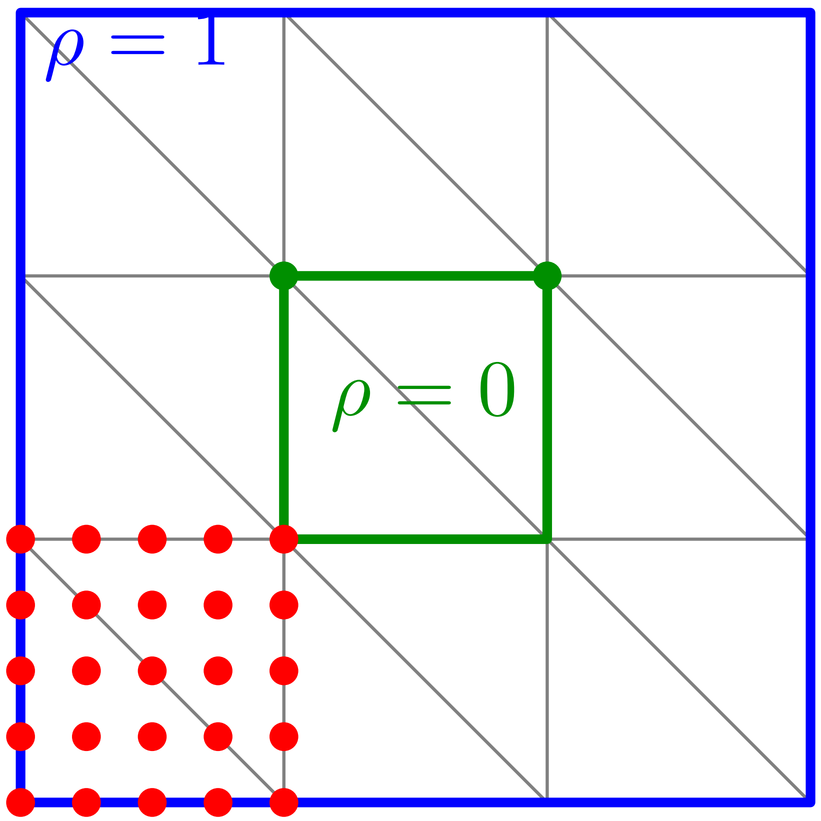




$\|G(\psi^3) - \nu\|_1$
$\simeq 10^{-9}$

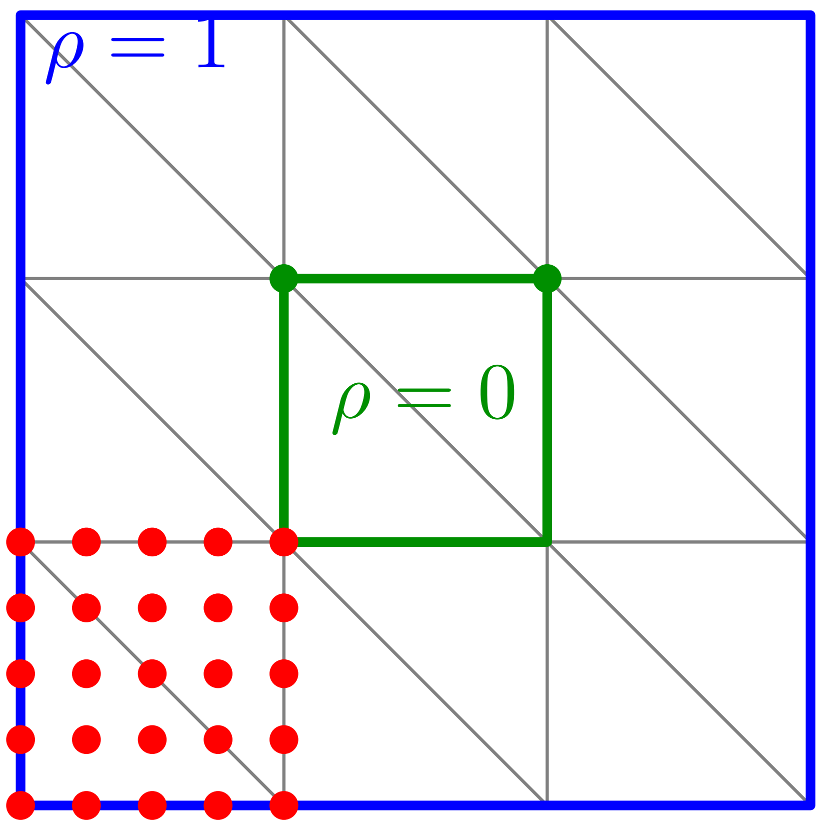
Quadratic cost: numerics



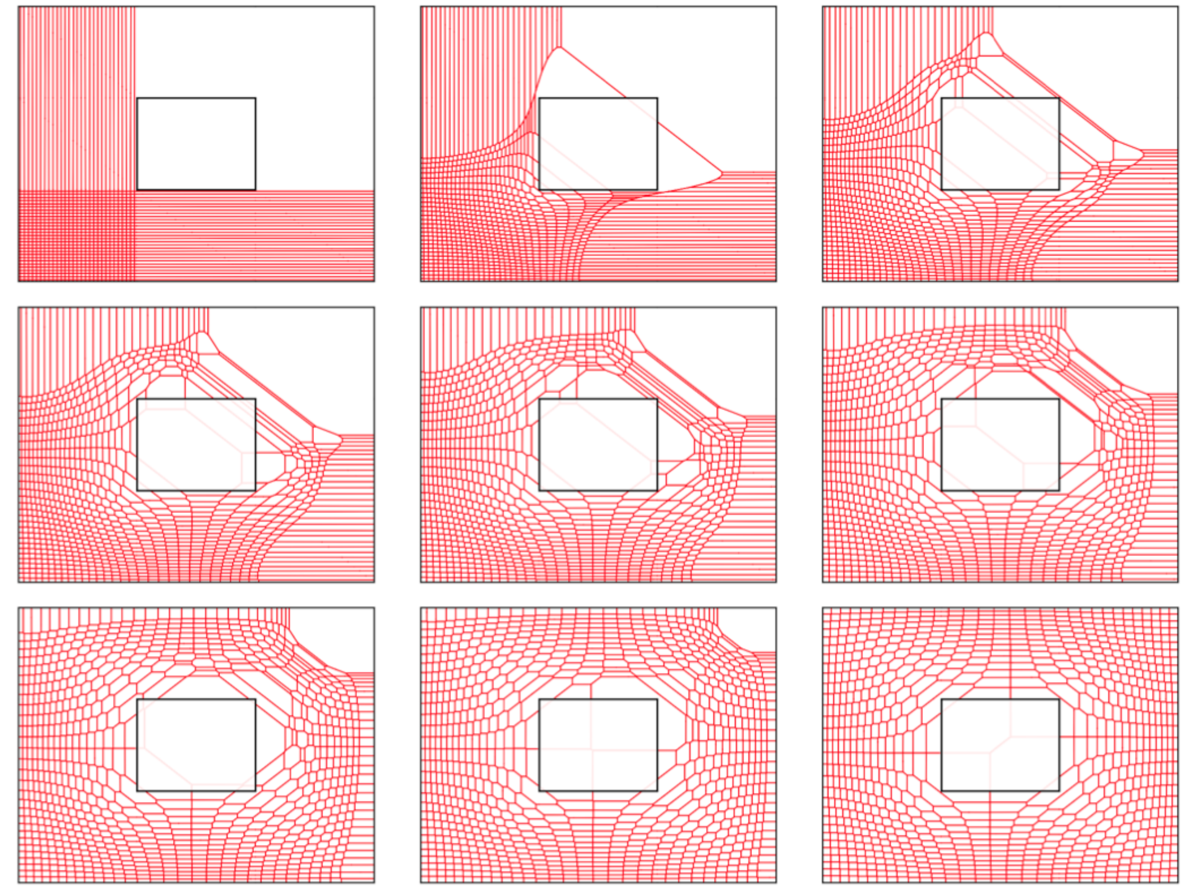
Source: PL density on $X = [0, 3]^2$

Target: Uniform grid Y in $[0, 1]^2$.

Quadratic cost: numerics



$\mathrm{Lag}(\psi^0)$



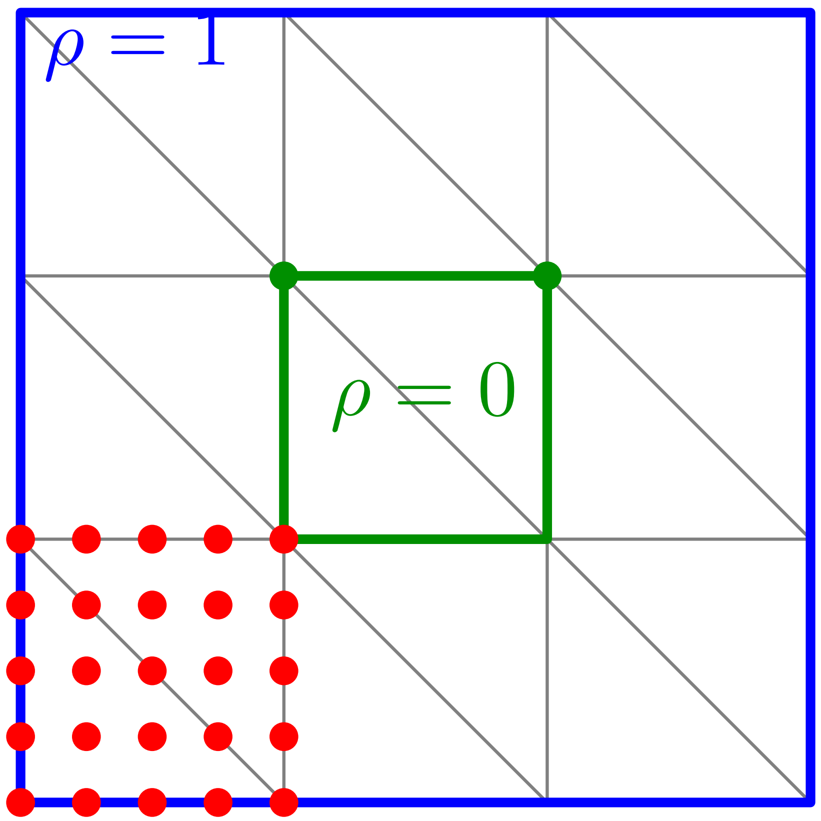
$\mathrm{Lag}(\psi^8)$

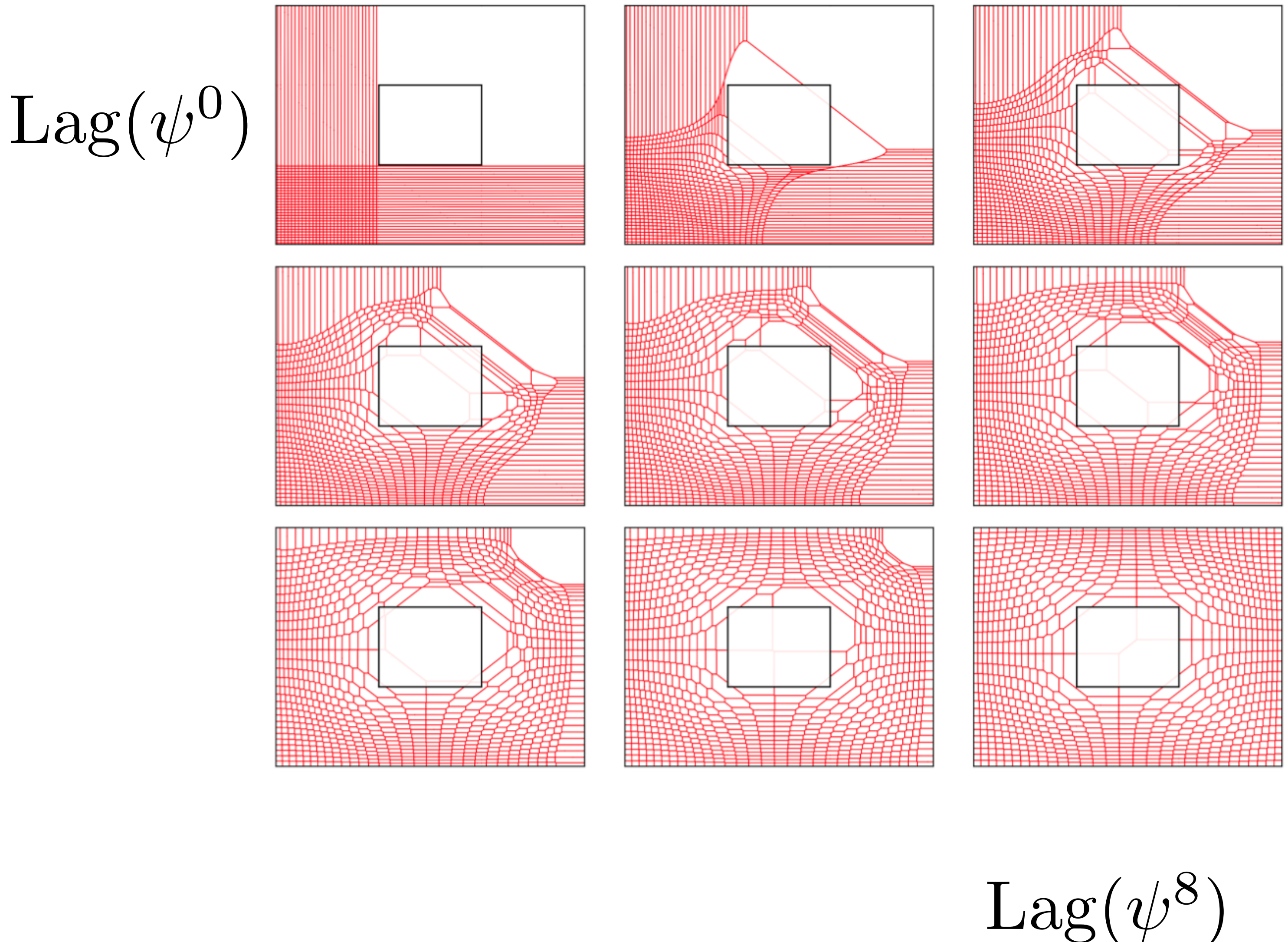
Source: PL density on $X = [0,3]^2$

Target: Uniform grid Y in $[0,1]^2$.

- The damped Newton's algorithm converges even when ρ vanishes.

Quadratic cost: numerics





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- The damped Newton's algorithm converges even when ρ vanishes.
- $N = 10^7$ pb solved in 17 iterations.psdot (python); geogram

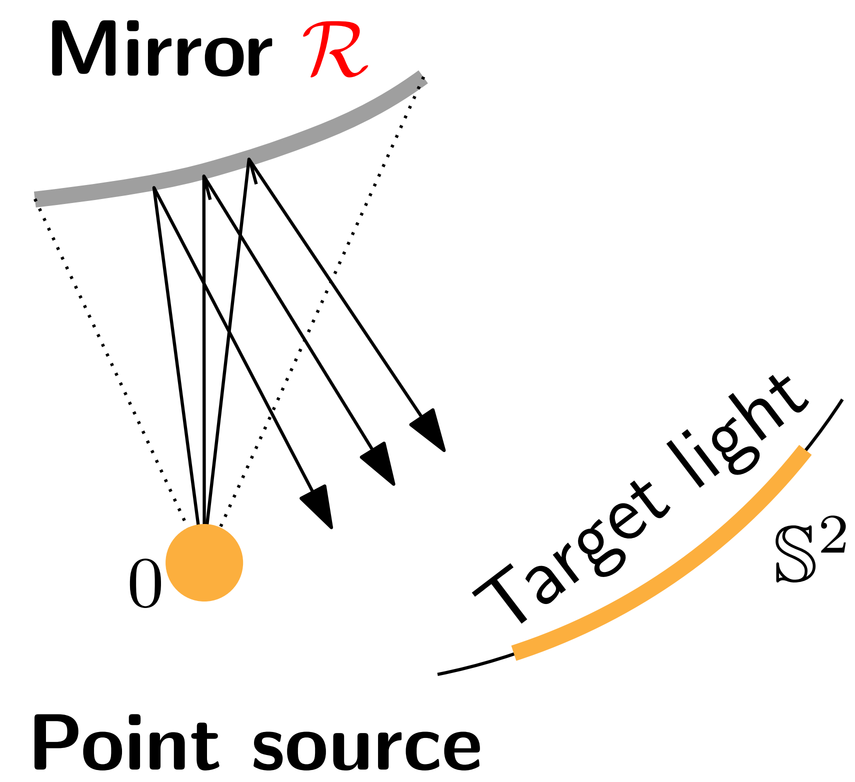
Outline

- Case 1: mirror for point light source
- Case 2: mirror for collimated source light

- Optimal transport
- Semi-discrete optimal transport
- Damped Newton algorithm

- Non-imaging optics: Far-Field target
- Non-imaging optics: Near-Field target

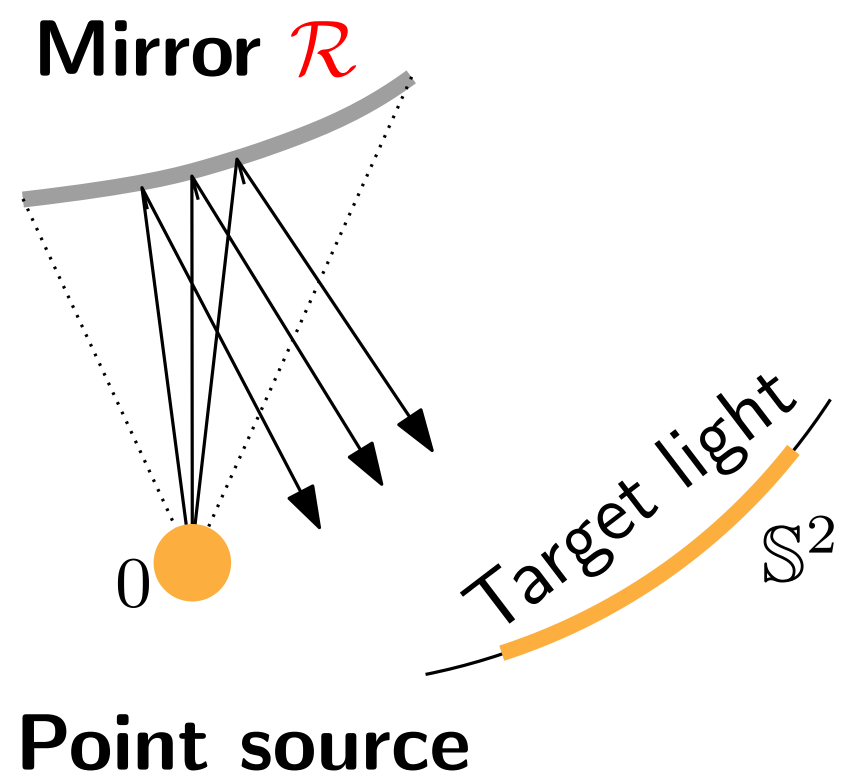
Mirror / Point light source: implementation



Mirror / Point light source: implementation

- **Newton schemes:**

Computation of descent direction / time step



Mirror / Point light source: implementation

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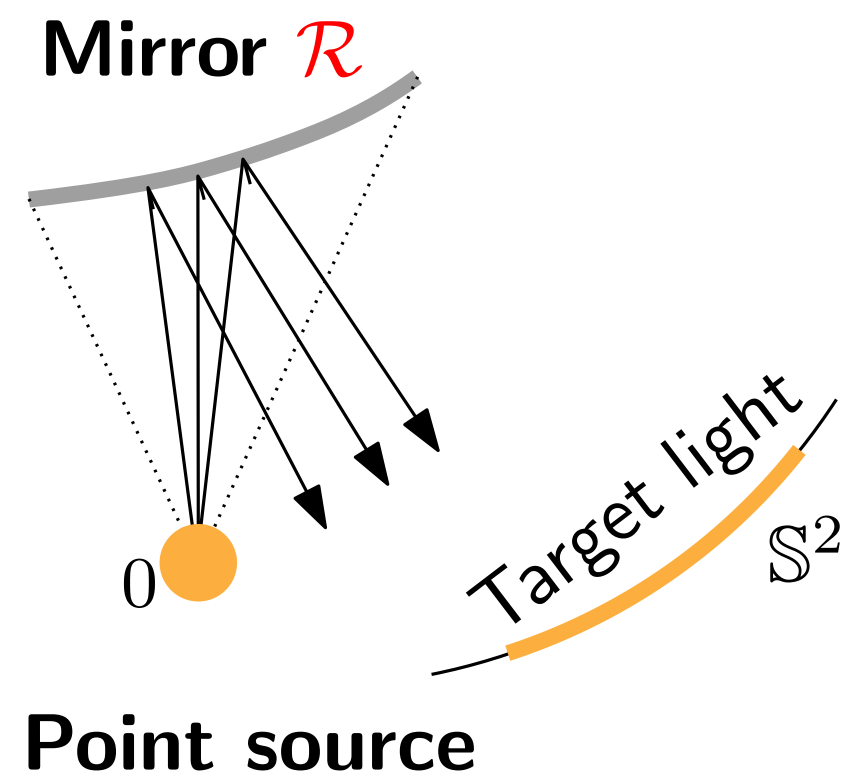
$$\int_{V_i} \mathrm{d}\,\mu(x) \qquad \int_{V_{ij}} \mathrm{d}\,\mu(x)$$



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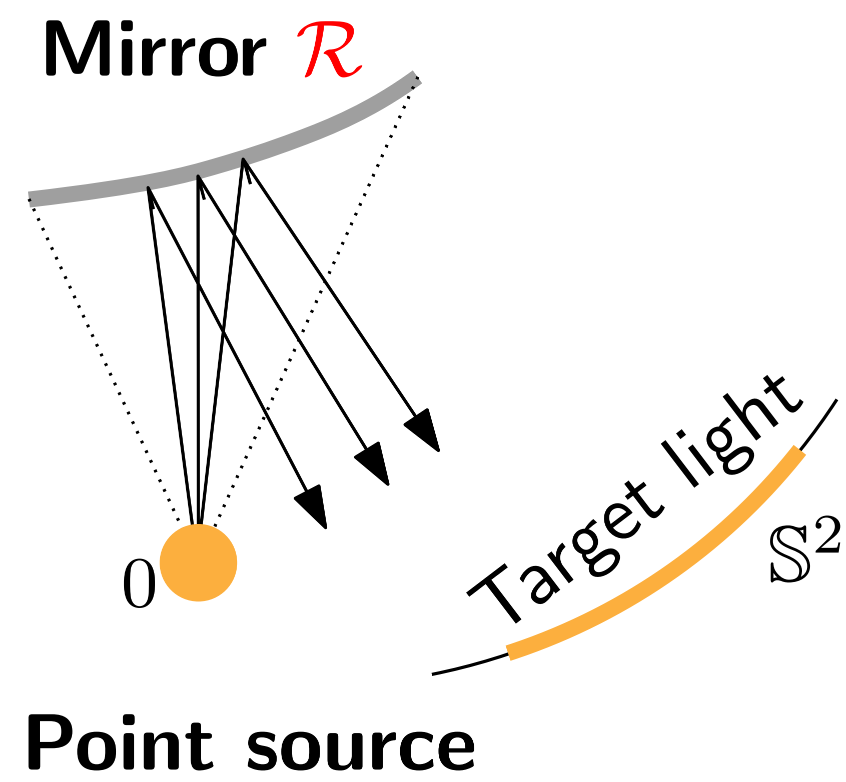

Main difficulty: computation of visibility cells V_i

Mirror / Point light source: implementation

Computation of Visibility (Laguerre) cells

Definition: Given $P = \{p_i\}_{1 \leq i \leq N} \subseteq \mathbb{R}^d$ and $(\omega_i)_{1 \leq i \leq N} \in \mathbb{R}^N$

$$\mathrm{Pow}_P^\omega(p_i) := \{x \in \mathbb{R}^d; i = \arg\min_j \|x - p_j\|^2 + \omega_j\}$$

Mirror / Point light source: implementation

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- Efficient computation of $(\mathrm{Pow}_P^\omega(p_i))_i$ using **CGAL** ($d = 2, 3$)

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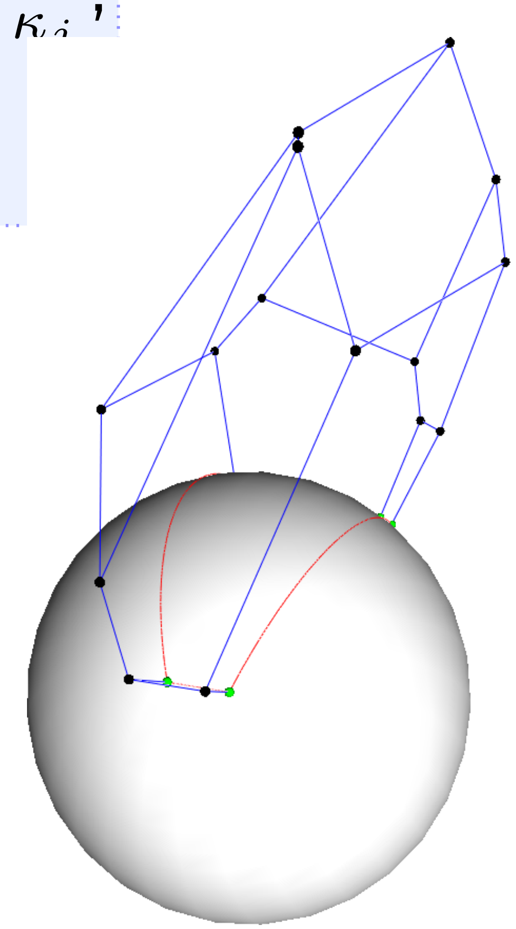
Lemma: With $\vec{\psi} = \log(\vec{\kappa})$, $p_i := -\frac{y_j}{2\kappa_j}$ and $\omega_i := -\|\frac{y_j}{2\kappa_j}\|^2 - \frac{1}{\kappa_j}$,

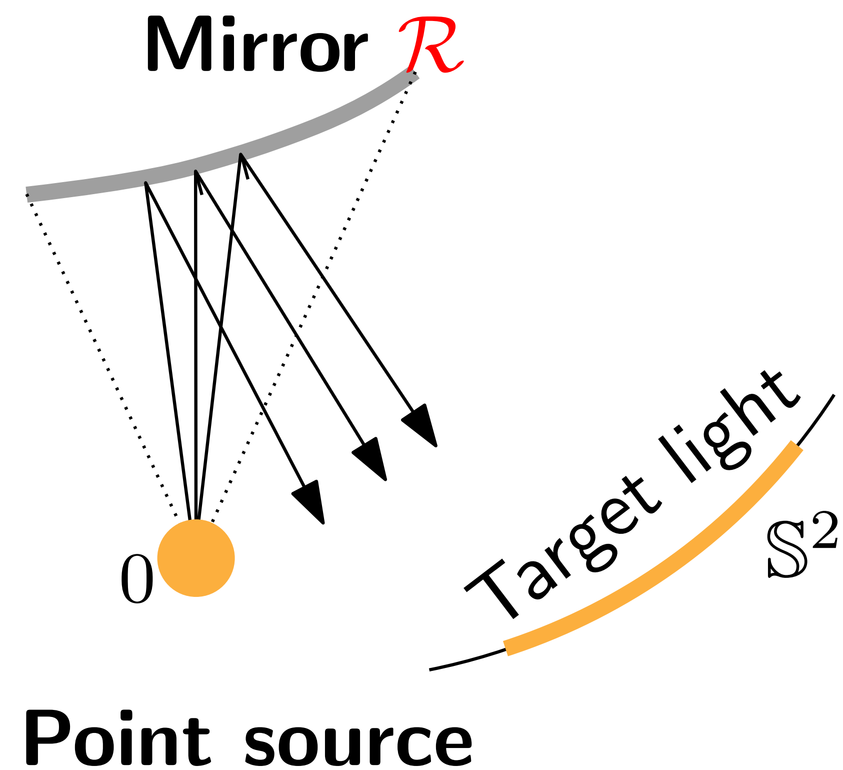
$$V_i(\kappa) = \mathrm{Pow}_P^\omega(p_i) \cap \mathbb{S}^2$$



Mirror / Point light source

$\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}$ discretization of Cameraman ($N = 400^2$).

μ = uniform measure on half-sphere $\mathbb{S}^2_+$



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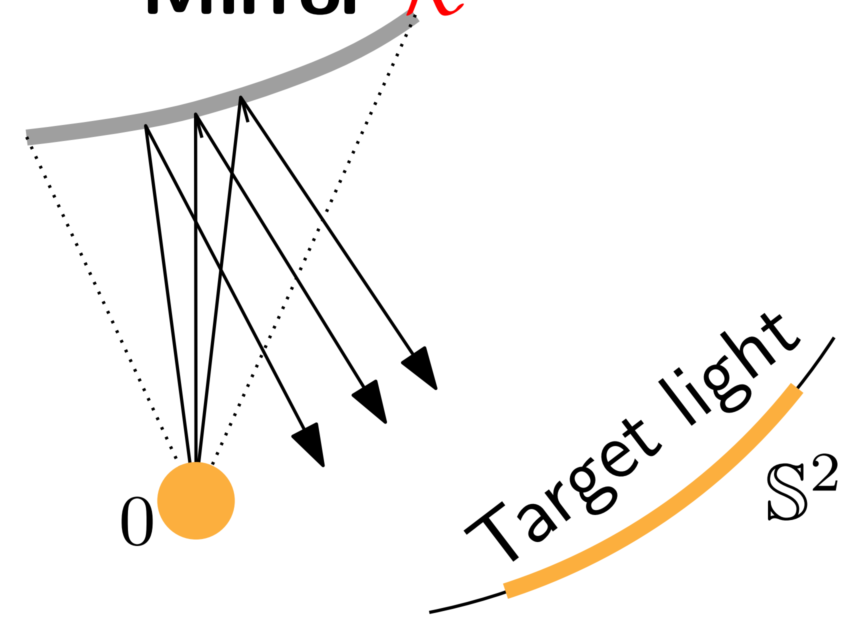
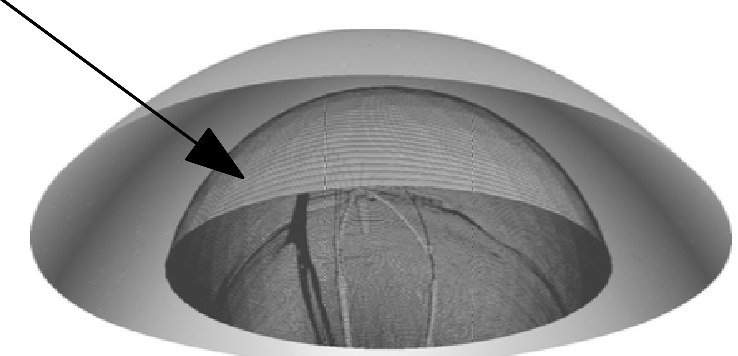
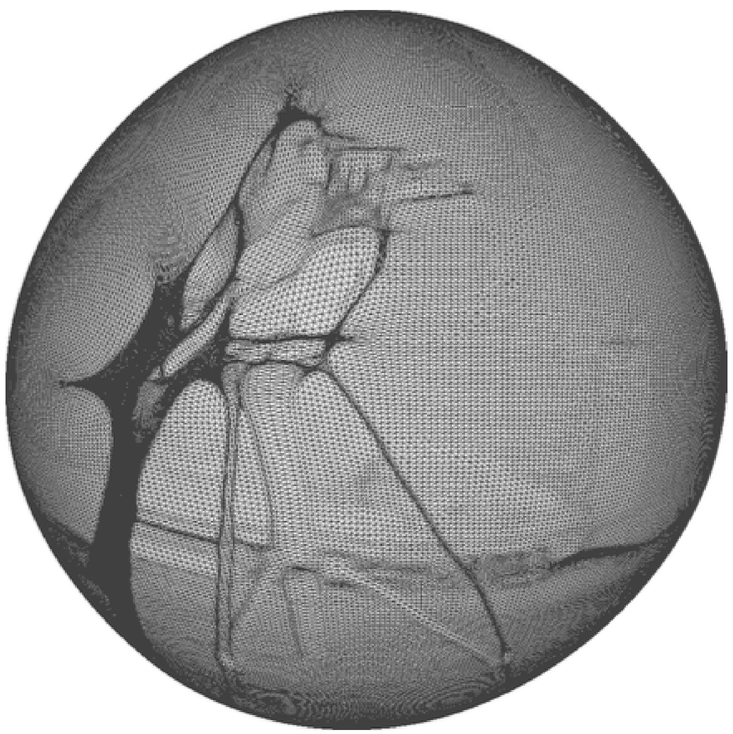


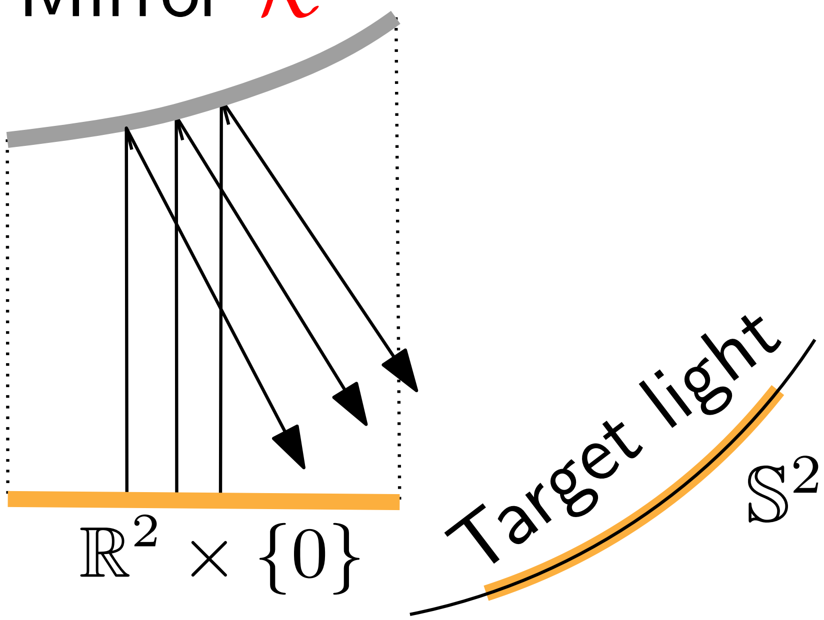
$\mu =$ uniform measure on half-sphere $\mathbb{S}^2_+$

$V_i(\psi) = \mathrm{Pow}(p_i) \cap \mathbb{S}^2$





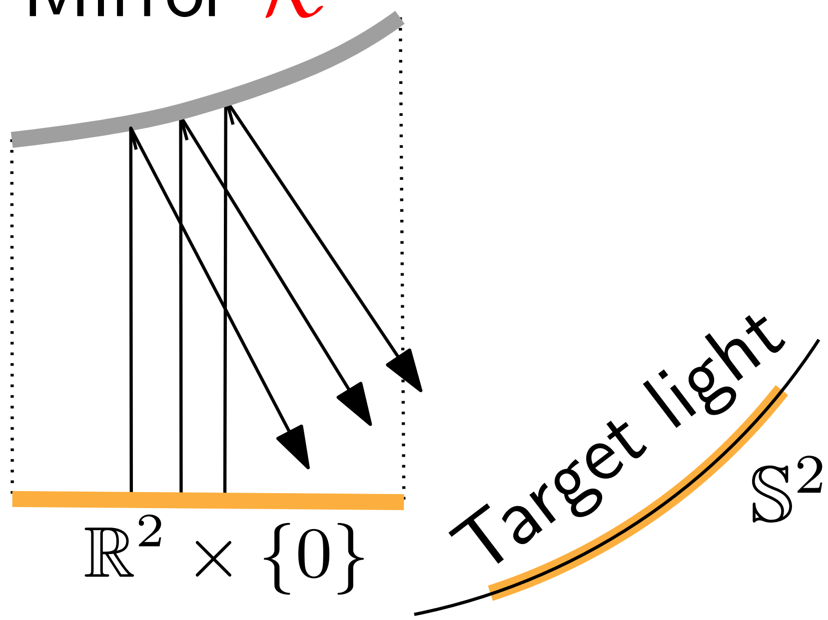
Collimated source / Far Field Target



Collimated source

targeted image $N = 400 \times 480$

Collimated source / Far Field Target



Collimated source

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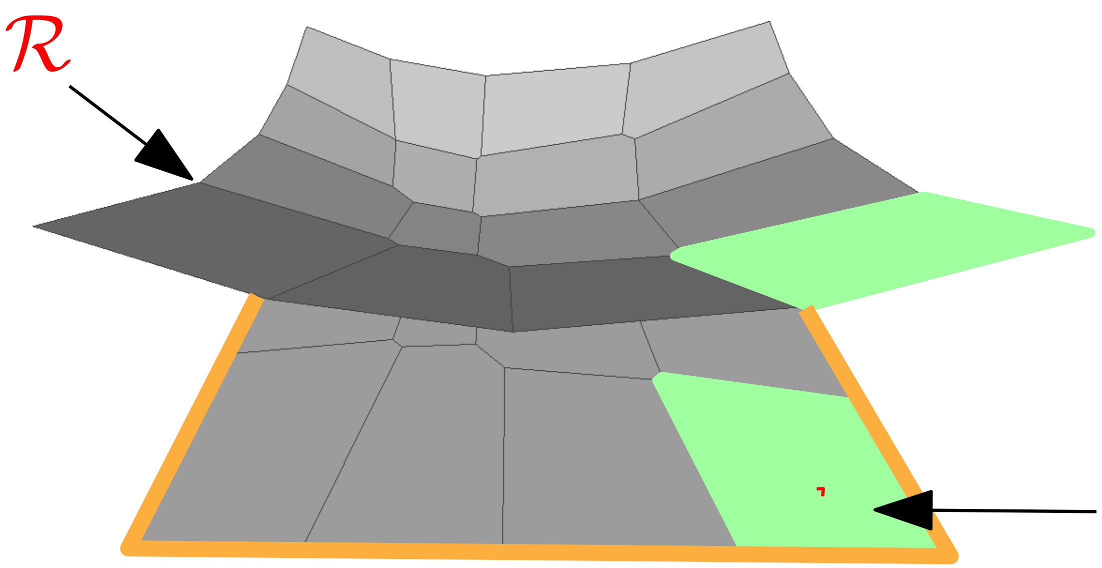

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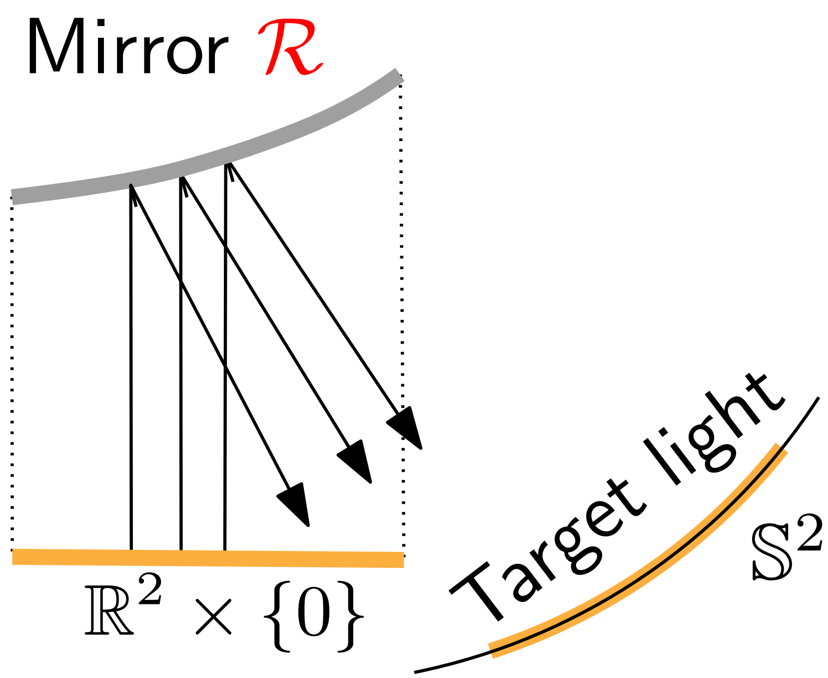

Collimated source

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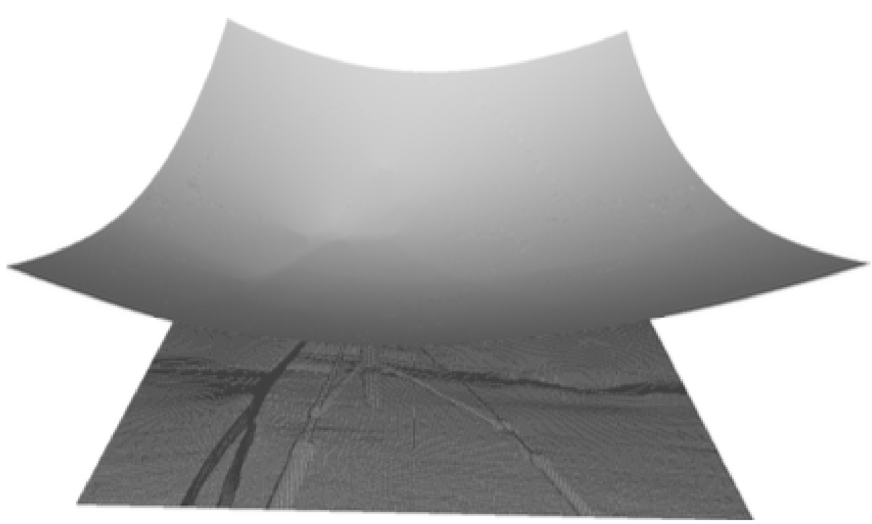
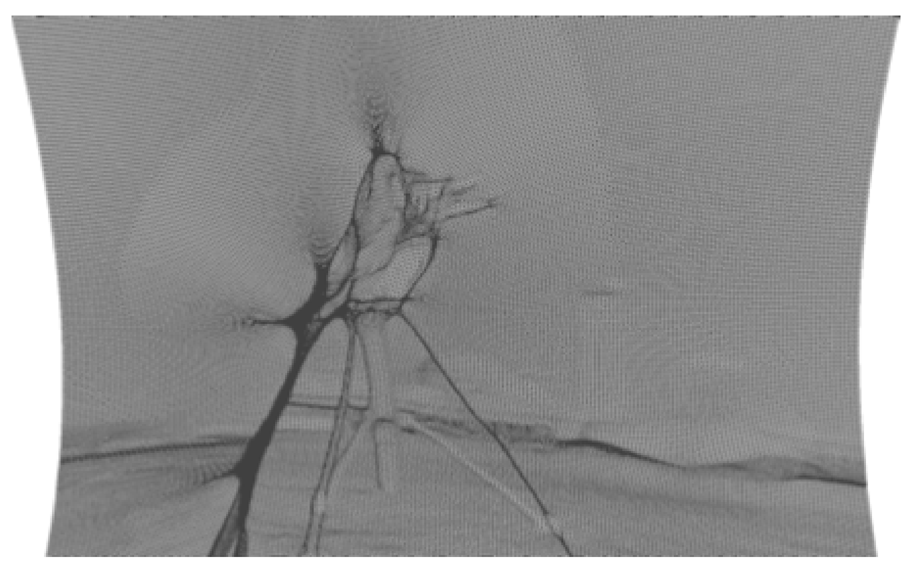
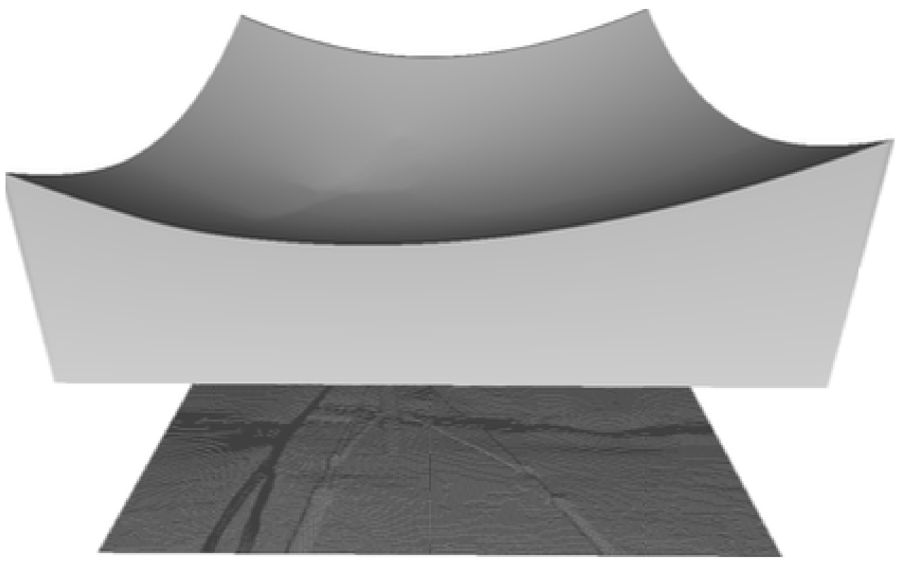
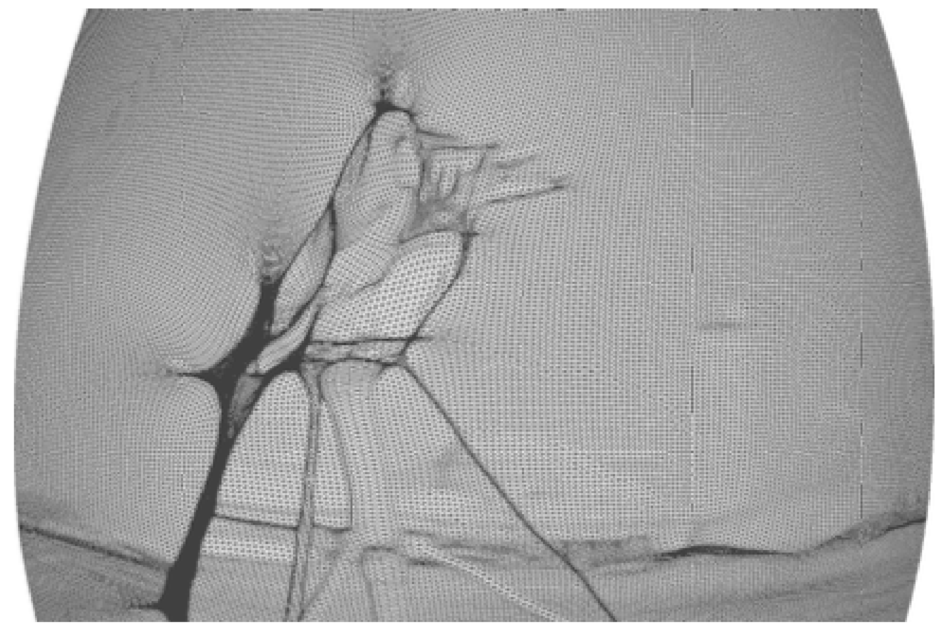
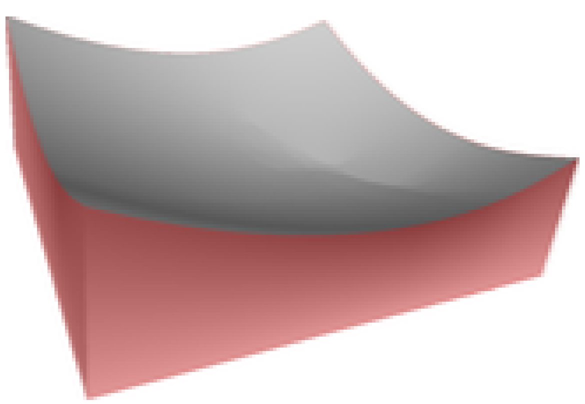


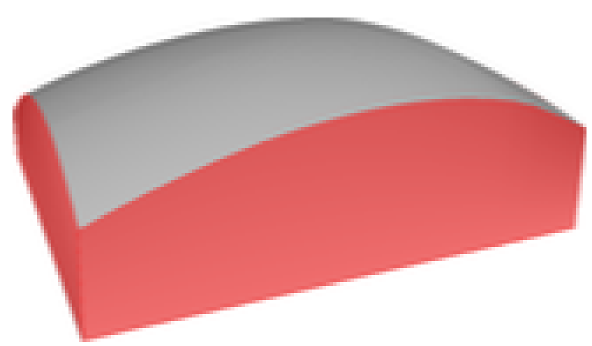


Lenses









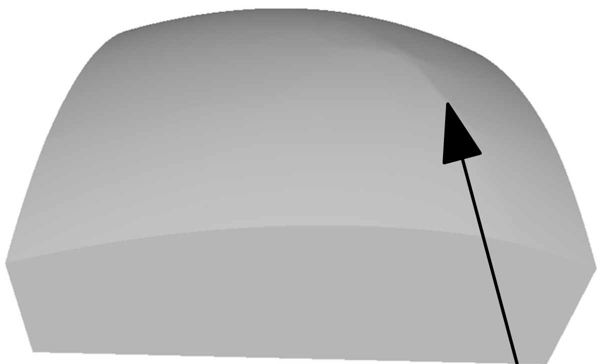
We solve 8 optical problems with one program

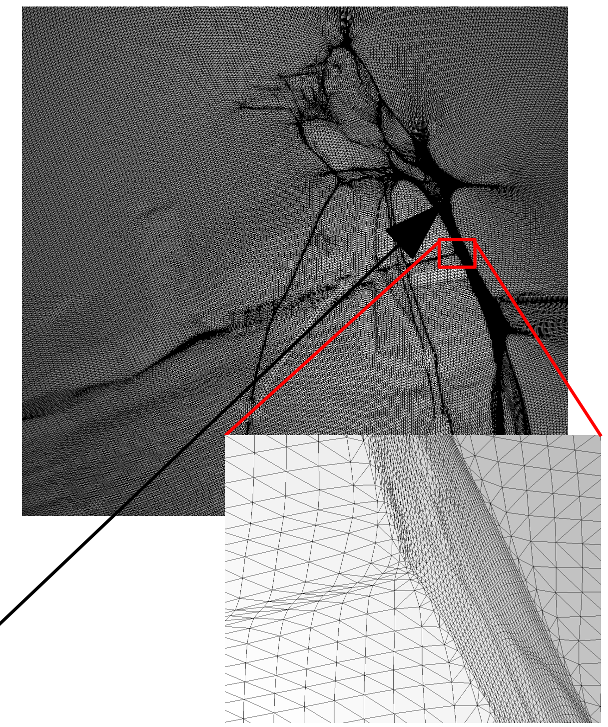
$\rightsquigarrow V_i(\psi) = \mathrm{Pow}(p_i) \cap X$ where $X = \mathbb{S}^2, \mathbb{R}^2 \times \{0\}$

$\rightsquigarrow$ Automatic differentiation

Collimated source / Far Field Target

mirror





mesh of the mirror

Singularity

Image rendered with LuxRender

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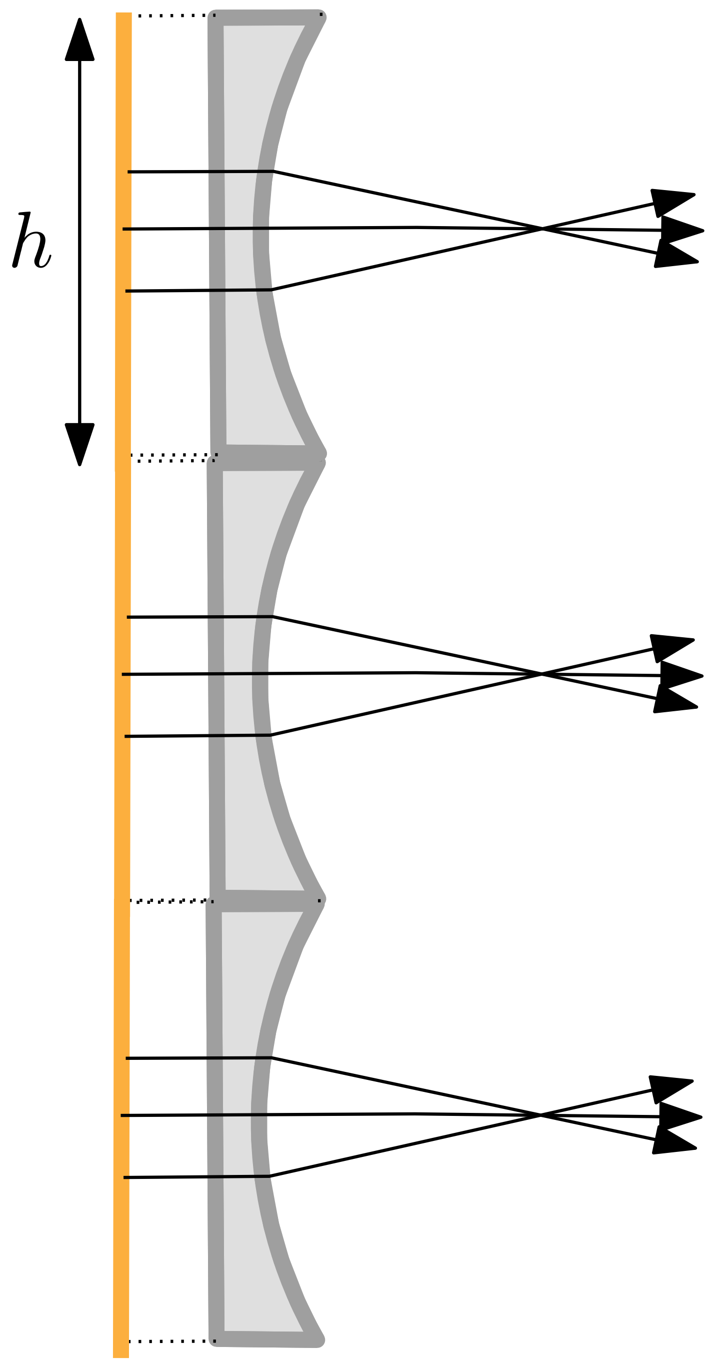
Problem with far-field assumption

Putting three copies of the same lens shifted by h...



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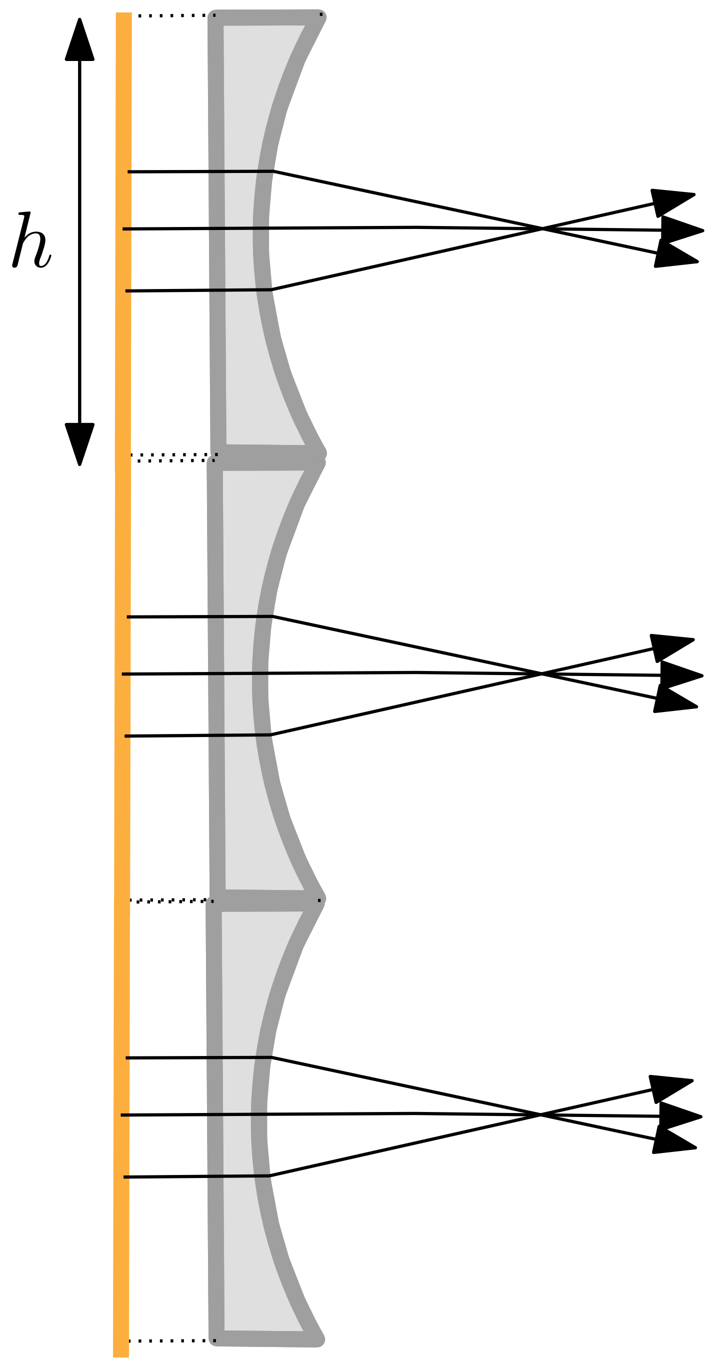

... produces a superposition of images shifted by h.

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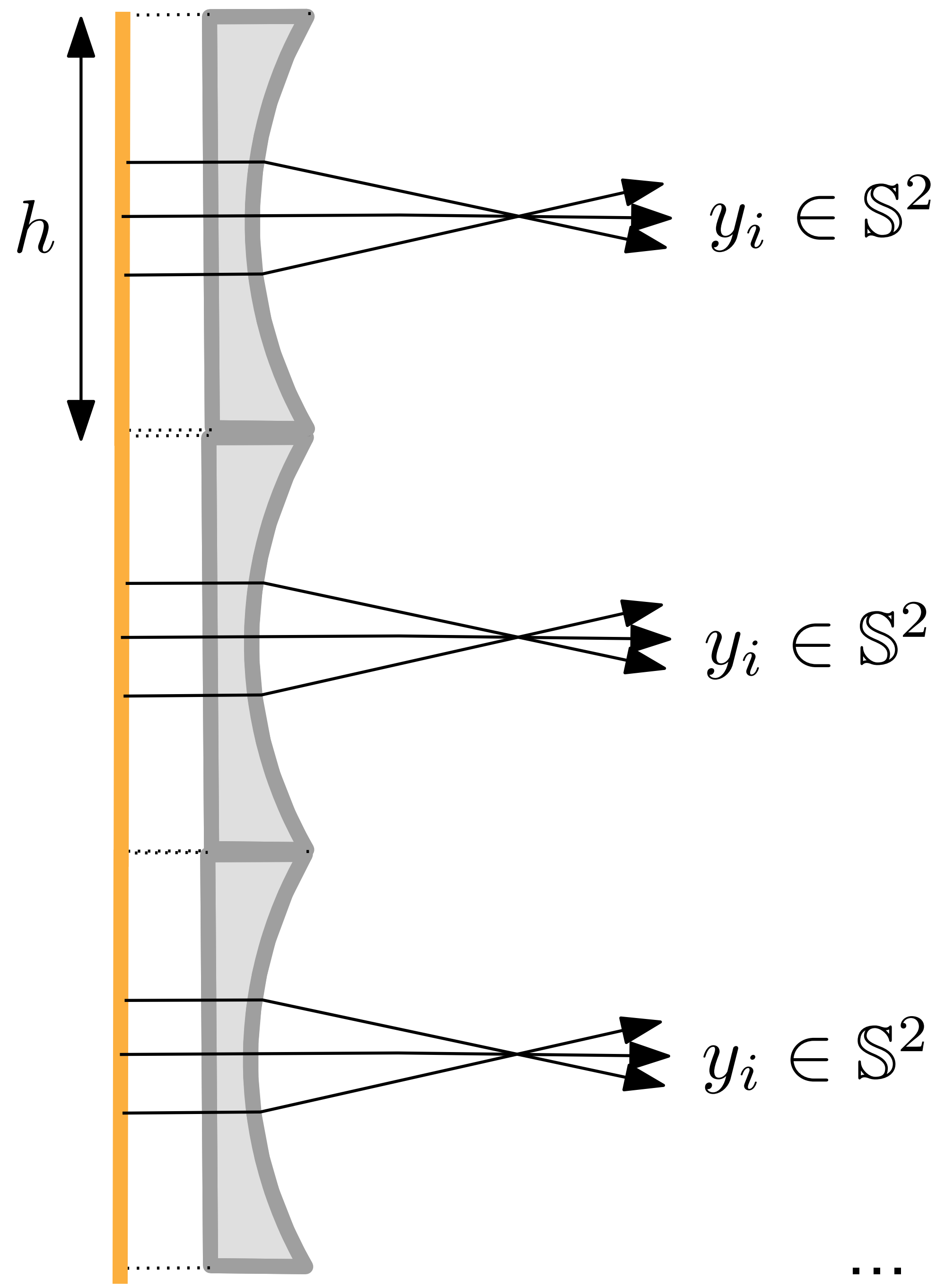

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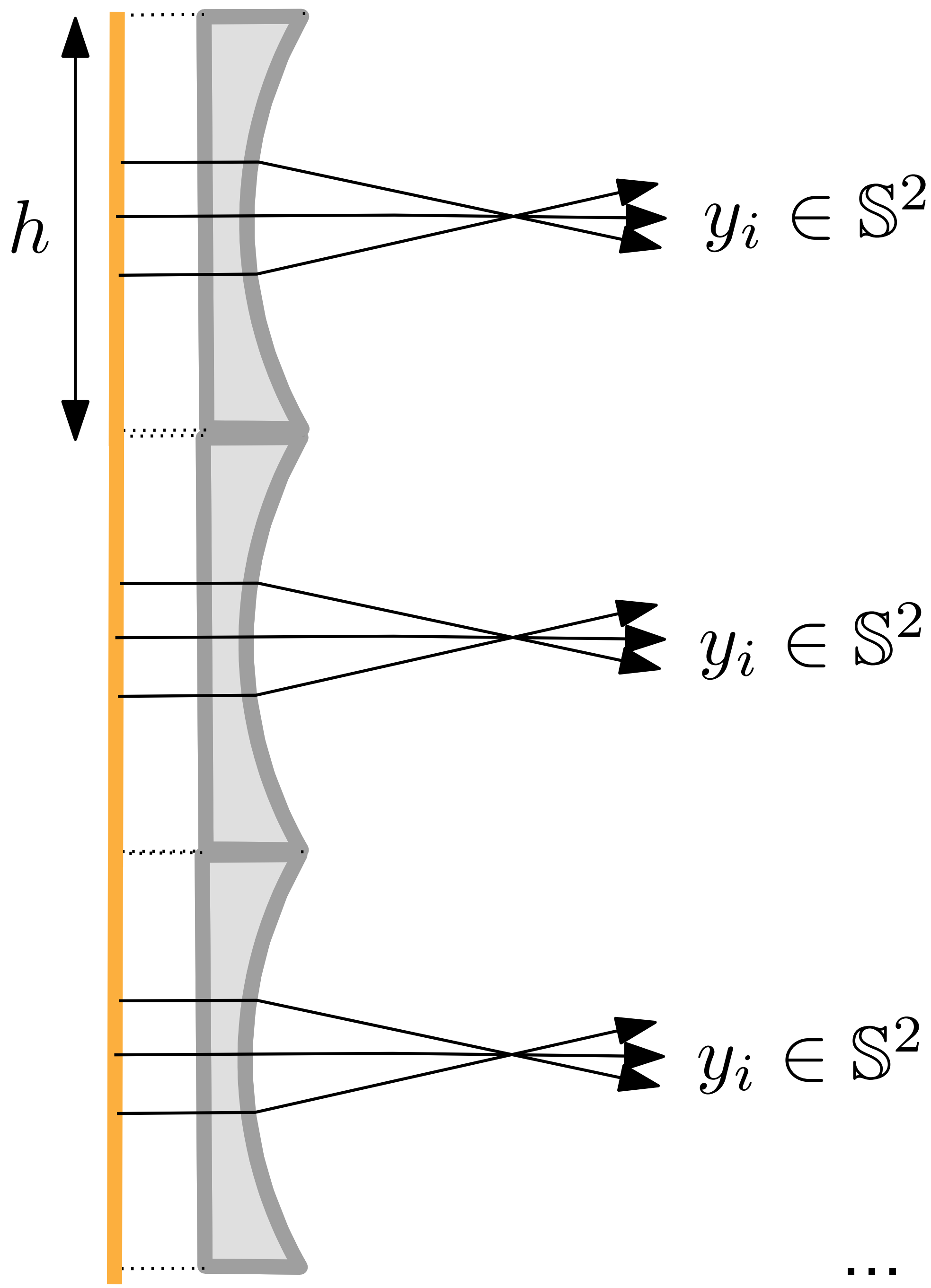


... produces a superposition of images shifted by h.

$\boxed{\text{One wants to produce images at finite distance} \longrightarrow \text{near-field problem.}}$

Iterated FF problem

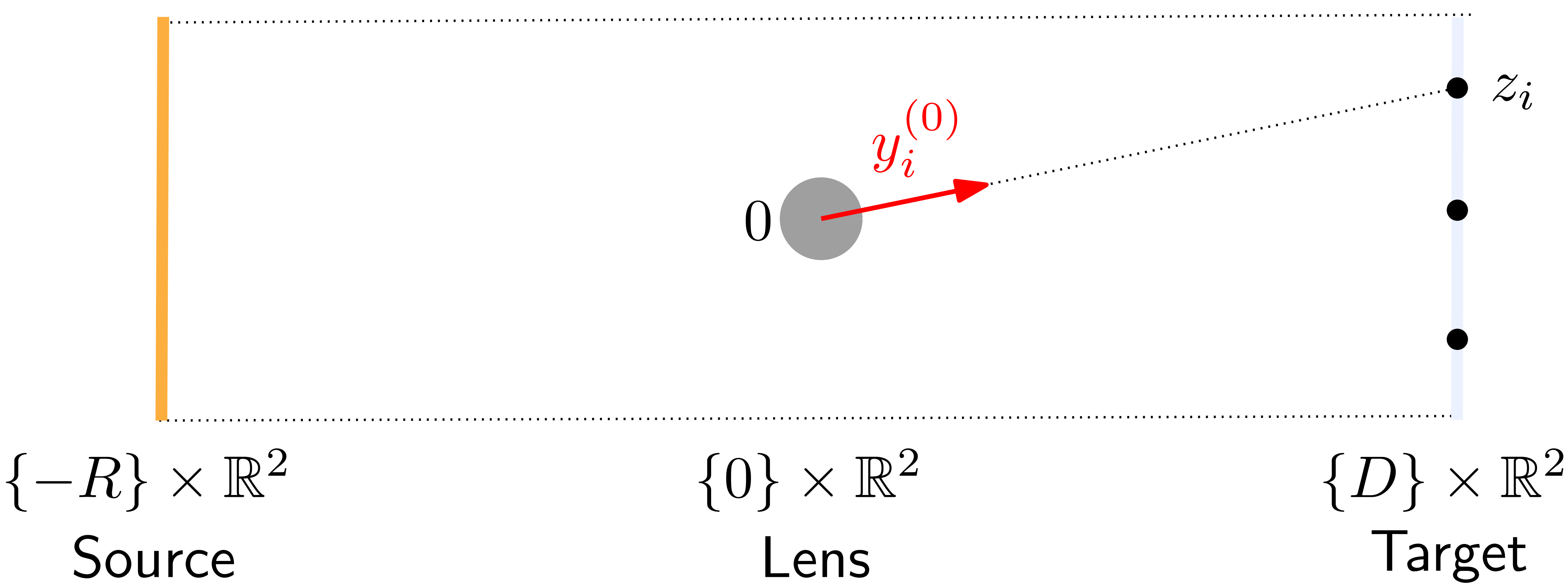
NF pb: Build a component $\mathcal{R}$ sending light towards $z_1, \ldots, z_N \in \{D\} \times \mathbb{R}^2$ (instead of $y_1, \ldots, y_N \in \mathbb{S}^2$))

Iterated FF problem

NF pb: Build a component $\mathcal{R}$ sending light towards $z_1, \ldots, z_N \in \{D\} \times \mathbb{R}^2$

We approximate solutions to the NF problem using a sequence of FF pb.

Step 0: Solve far-field problem with target $y_i^{(0)} = z_i / \|z_i\|$



Iterated FF problem

NF pb: Build a component $\mathcal{R}$ sending light towards $z_1, \ldots, z_N \in \{D\} \times \mathbb{R}^2$

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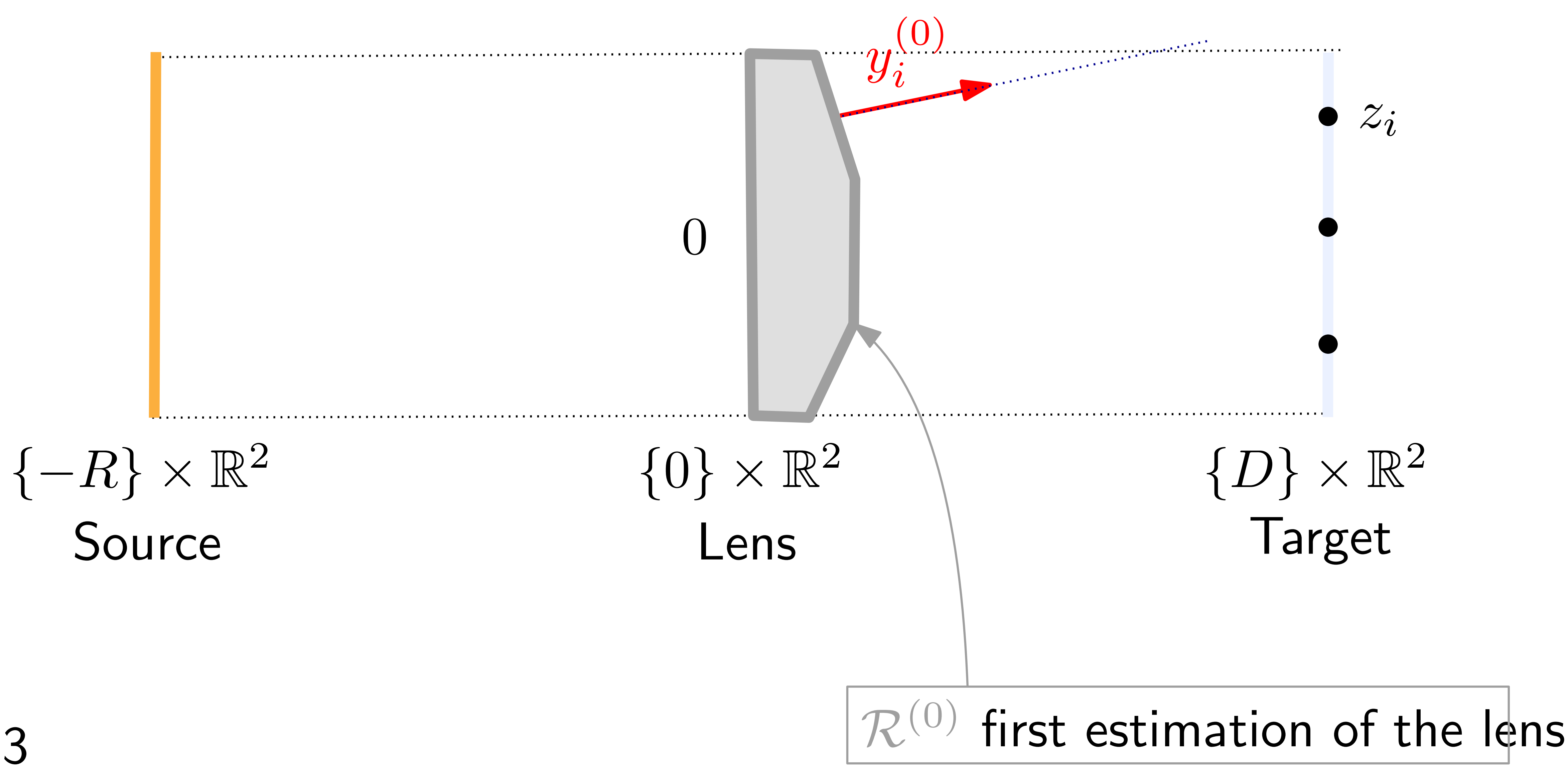

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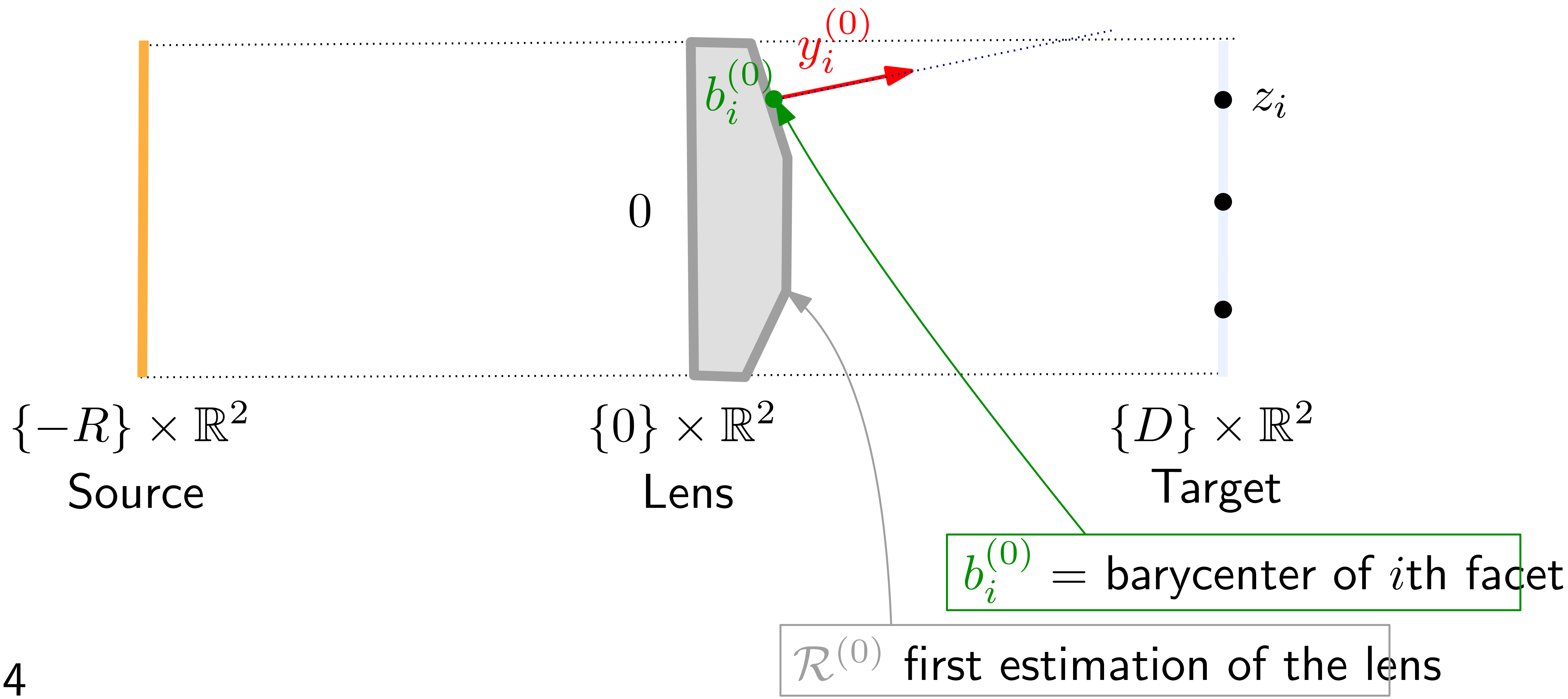

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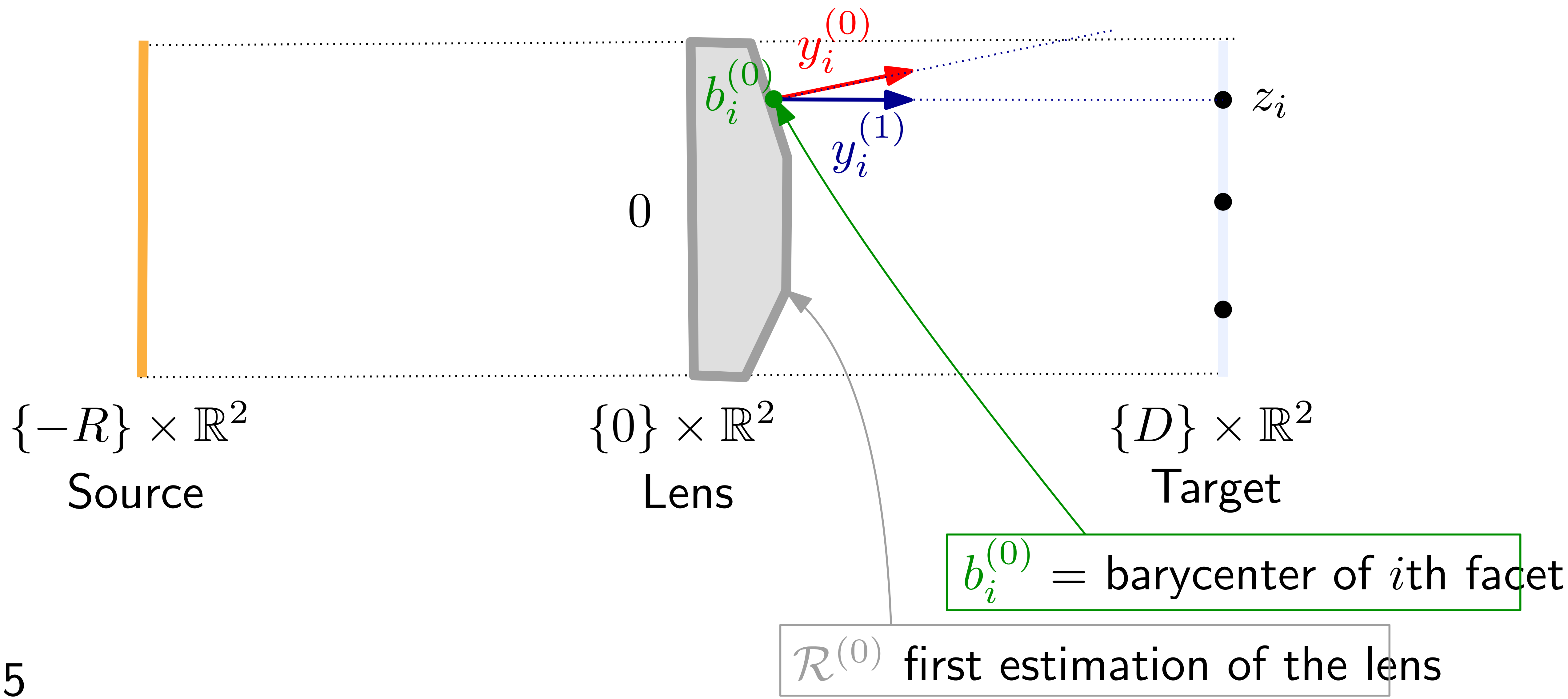

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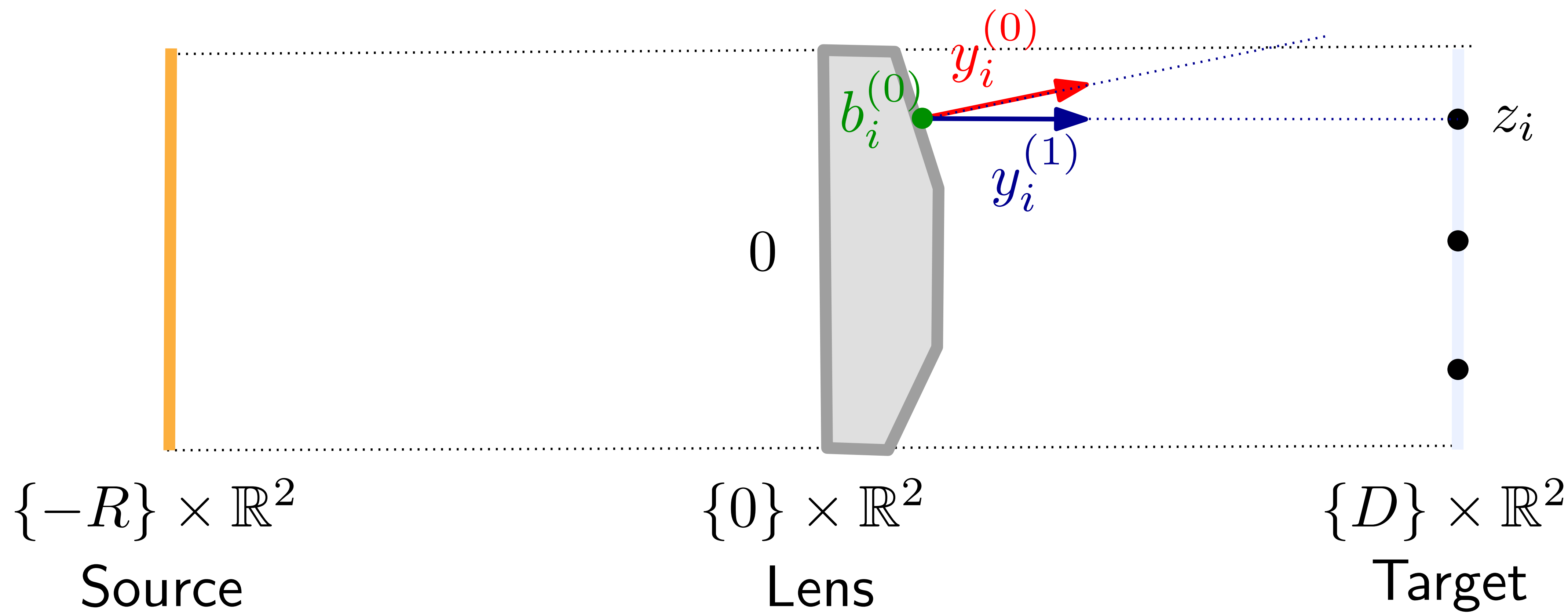


Step k+1: Solve far-field problem with target $y_i^{(k+1)} = (z_i - b_i^{(k)})/\|z_i - b_i^{(k)}\|$,

Efficient heuristic to solve NF problem using a FF solver...

Convergence of the algorithm

Target

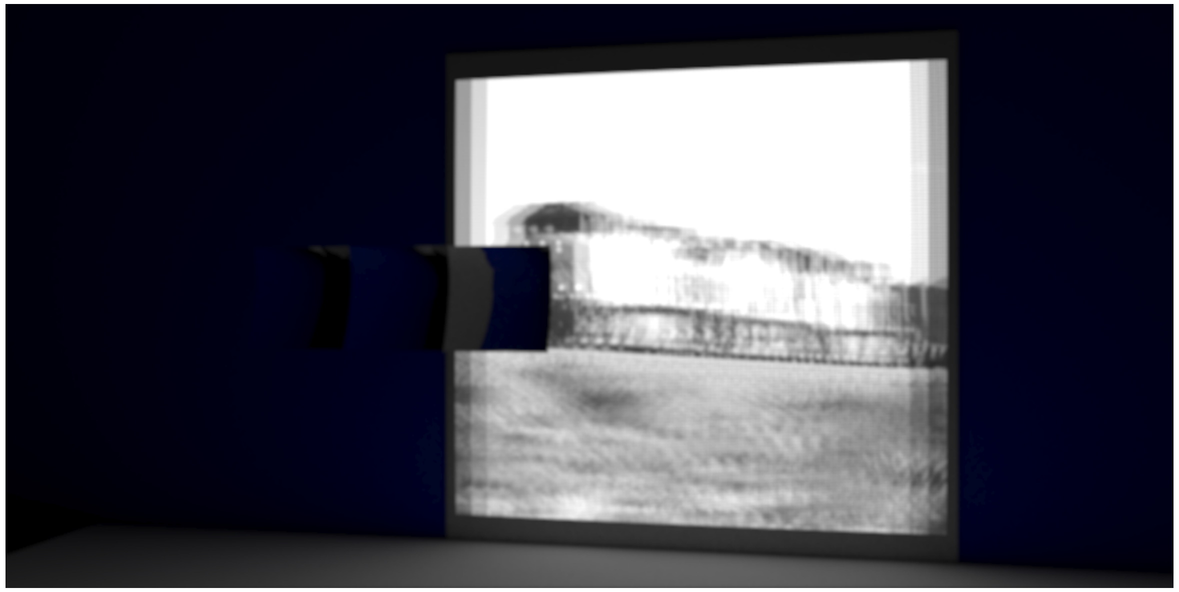
1st iteration

2nd iteration

5th iteration

size	$k=1$	$k=2$	$k=3$	$k=4$	**Total** $(k=6)$
128^2	9s	9s	6s	2s	31s
256^2	38s	61s	38s	31s	228s
512^2	245s	294s	240s	194s	1303s
1024^2	1598s	2095s	1586s	1489s	9077s

Pillows



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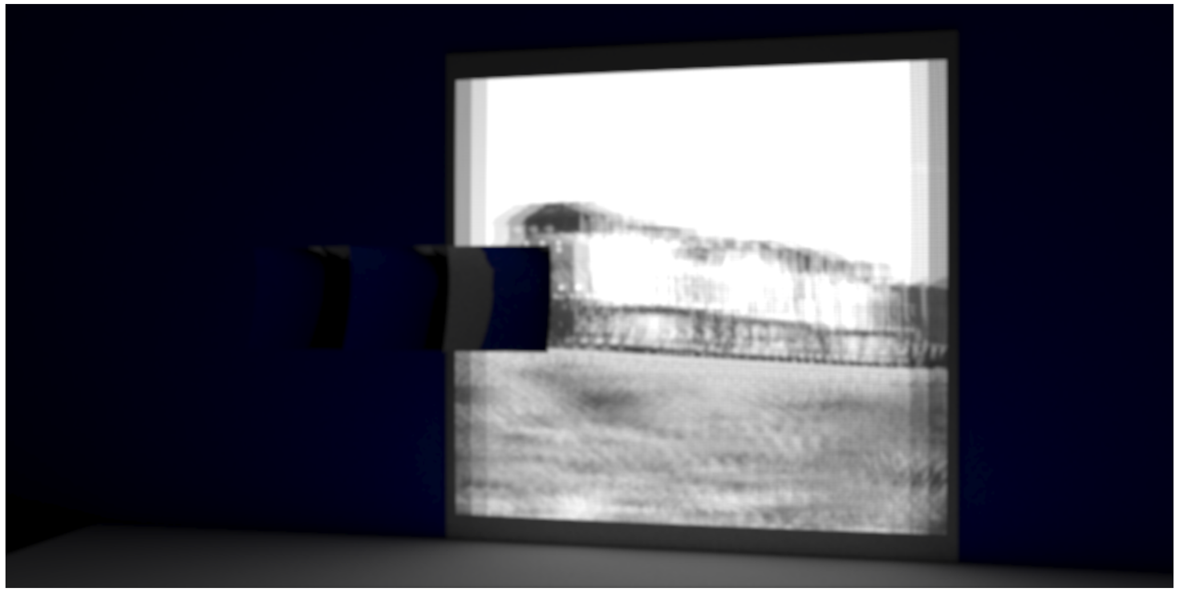

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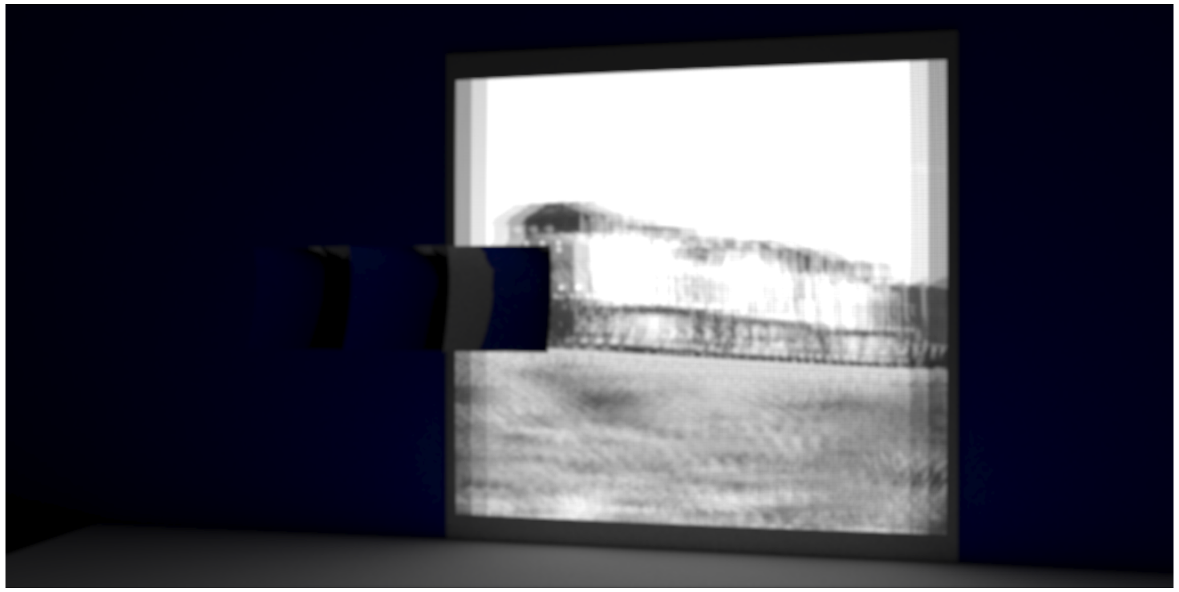
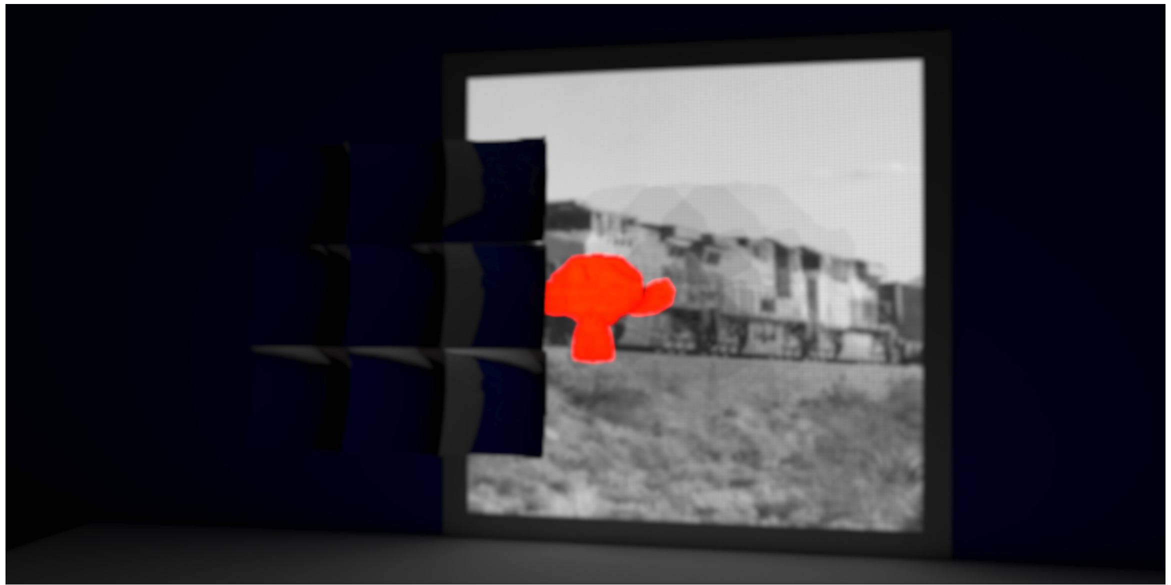
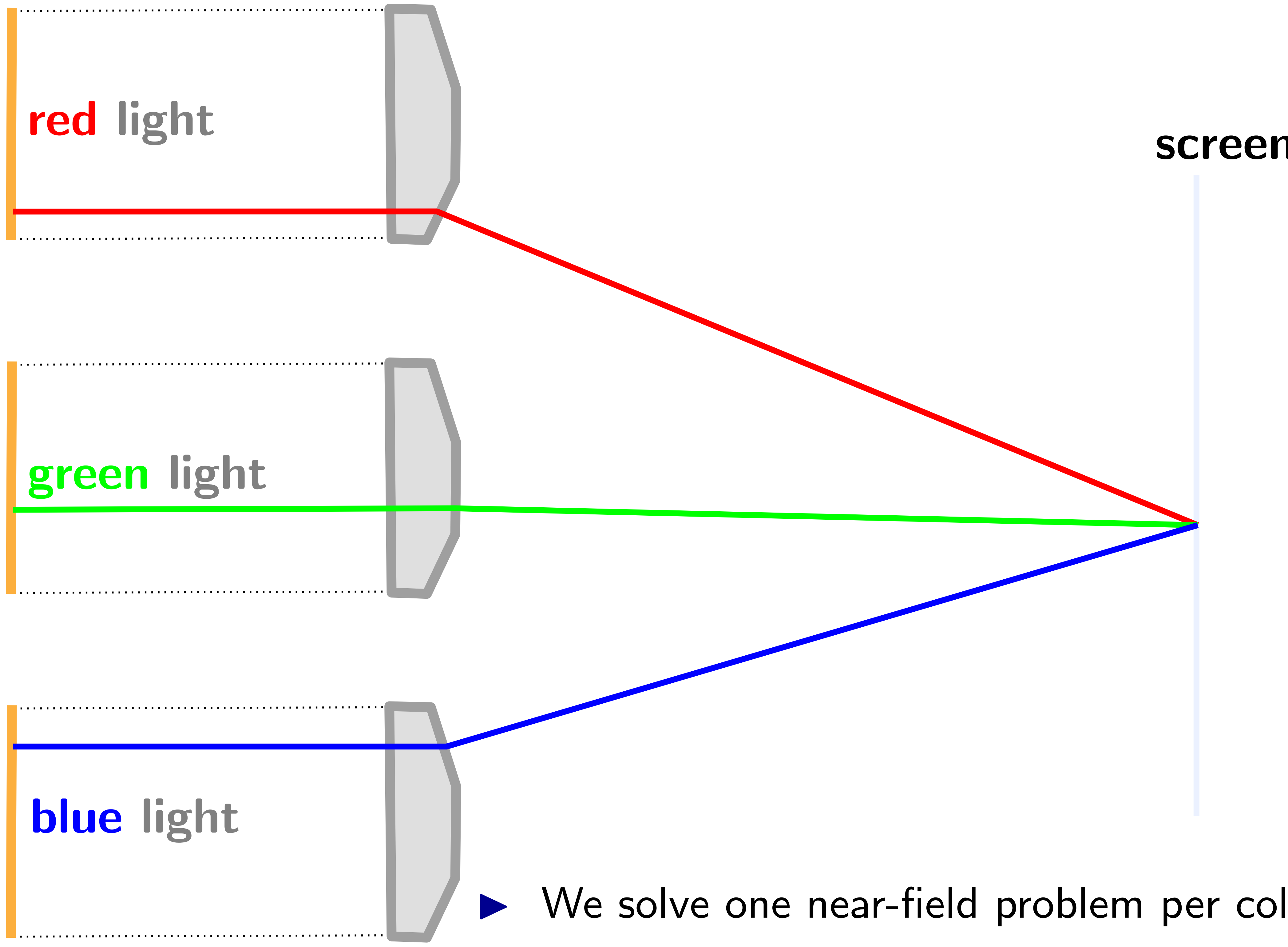


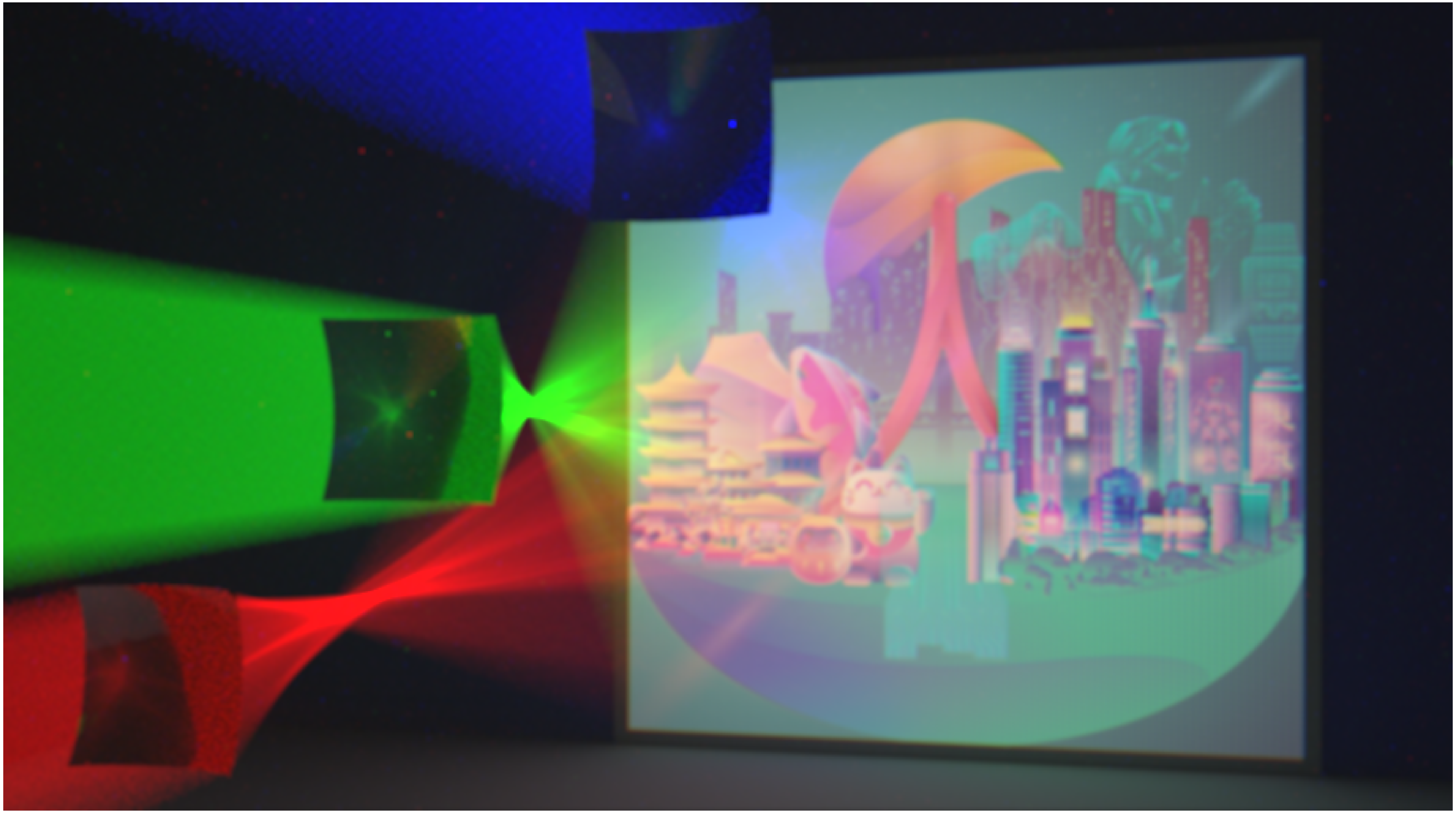


Color channels



- We solve one near-field problem per color channel.
- Near-field assumption needs to be taken into account for the image to be perfectly superimposed on the screen.

Color channels



Physical prototypes

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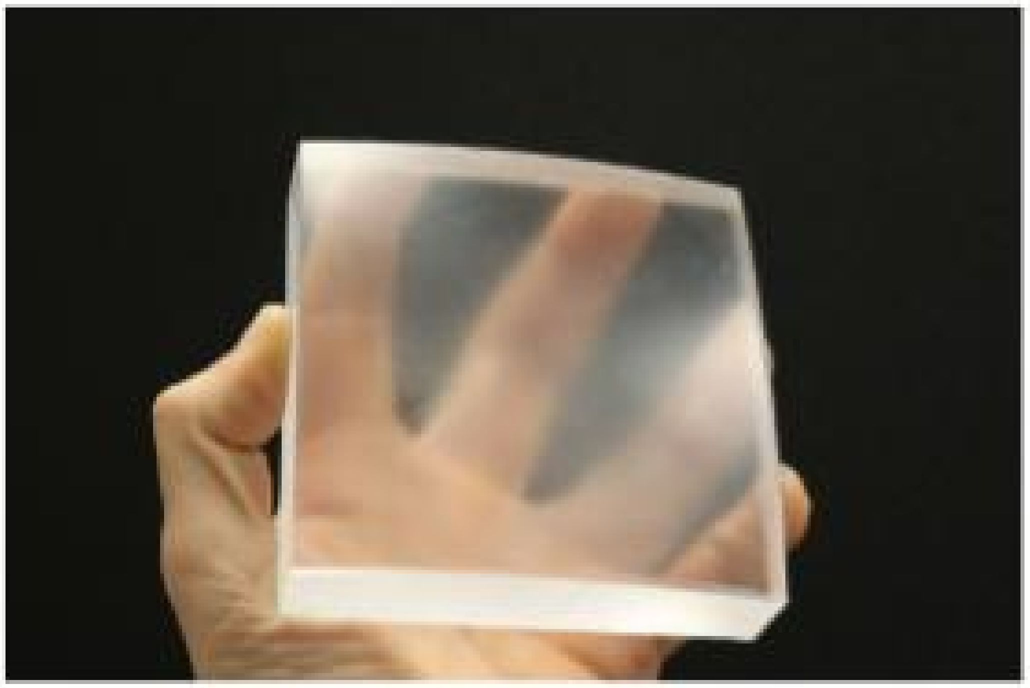
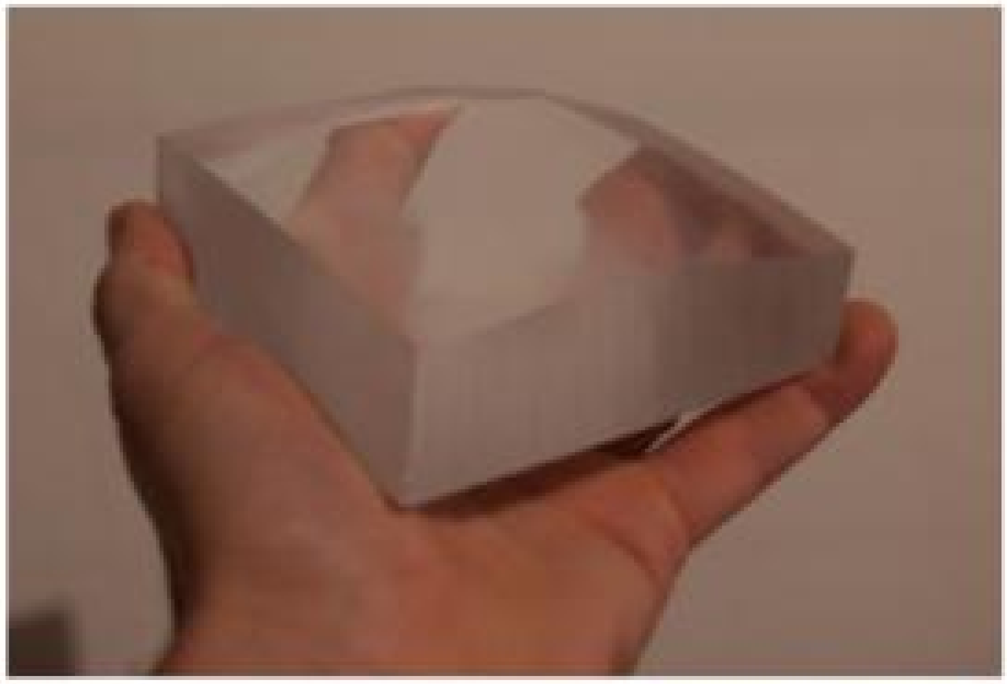

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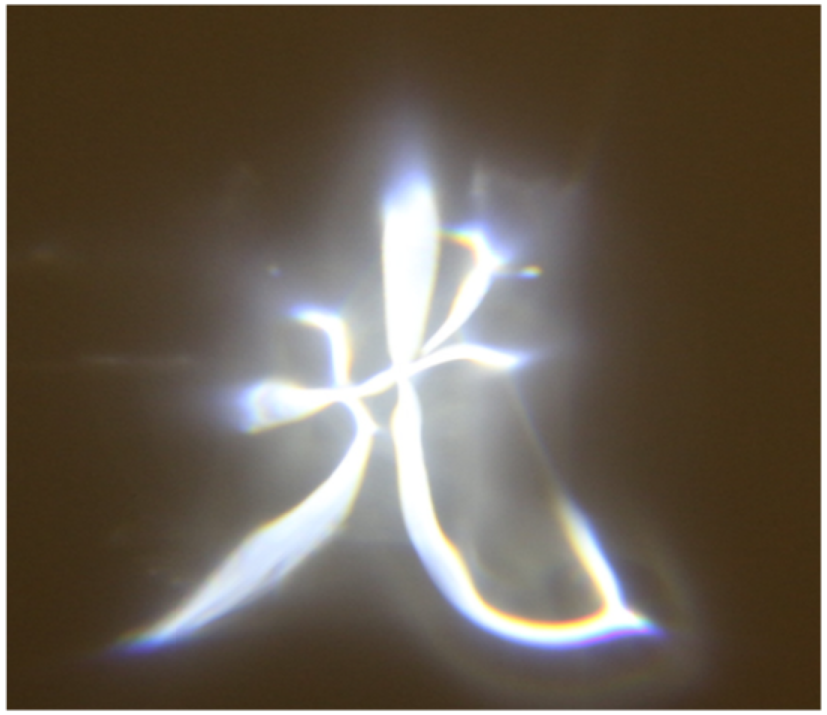

Conclusion

We solved 4 inverse problems arising in nonimaging optics
using semi-discrete approach and optimal transport

- Each problem is a Monge-Ampère equation
- For far-field target, OT problem on $\mathbb{R}^2$ or $\mathbb{S}^2$ $\rightsquigarrow$ Newton algorithm
- Iterative procedure for real-life light target

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Ongoing work

$\rightsquigarrow$ Generalization to generated jacobian equations (application to optics, near field target) : Anatole Gallouet's talk

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$\rightsquigarrow$ Metasurfaces (with Cristian Gutierrez)

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Thank you!