A maximum-flow model for digital elastica shape optimization

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Outline

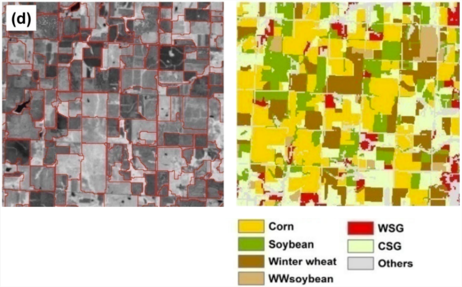
Motivation

Image analysis

The problems we are interested in come from *image analysis*.

Segmentation **Restoration** **Inpainting**

X. Li, Zhao, Han, Tong, and Yang 2019



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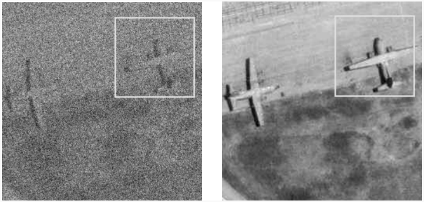
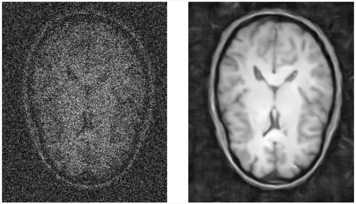


Xu et al. 2018



Jiang et al. 2018

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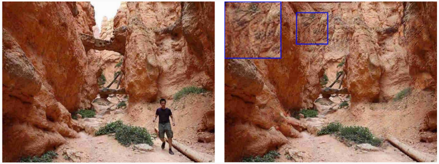
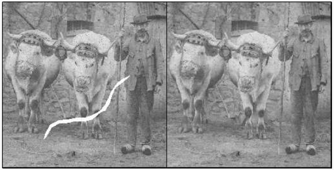


Yu et al. 2018



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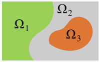
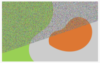


Restoration: $f_{\widehat{\boldsymbol{I}}} = \arg\min_f E_{den}(f, f_{\widetilde{\boldsymbol{I}}})$.



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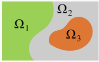

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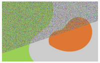

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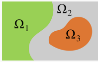

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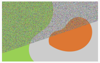

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We focused on *variational approaches* to solve these problems.
Energies are defined by terms that guide the optimization towards the solution of interest, e.g.,

- *Data fidelity*. The solution should not differ much from the input.
- *Spatial coherence*. Images are composed of regions with low variability in color.

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Geometric priors

The *Mumford Shah* (Mumford and Shah 1989) is a model for segmentation and denoising.

$$\min_{f,\mathcal{K}} \alpha \int_{\Omega} \|f_{\boldsymbol{I}} - f\|^2 dx + \beta \int_{\Omega \setminus \mathcal{K}} \|\nabla f\|^2 dx + \lambda Per(\mathcal{K}).$$

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In this thesis, we are interested in the combined use of *perimeter* and *squared curvature* as geometric priors.

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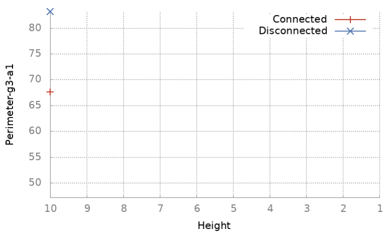
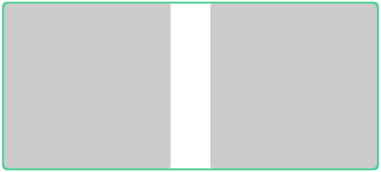
Completion property

$$\min_{X \subset \Omega} Data(X) + Perimeter(\partial X).$$





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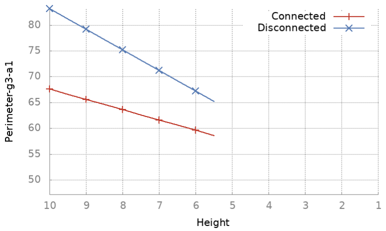

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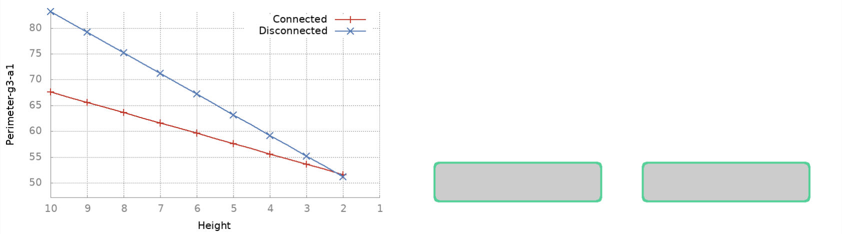

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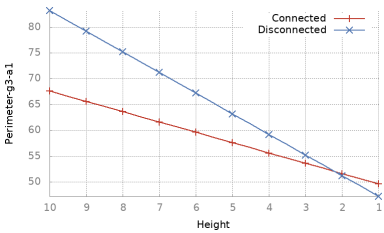

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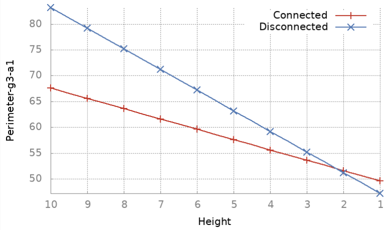

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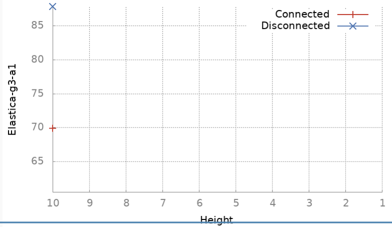

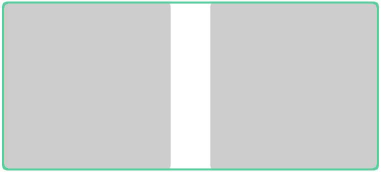

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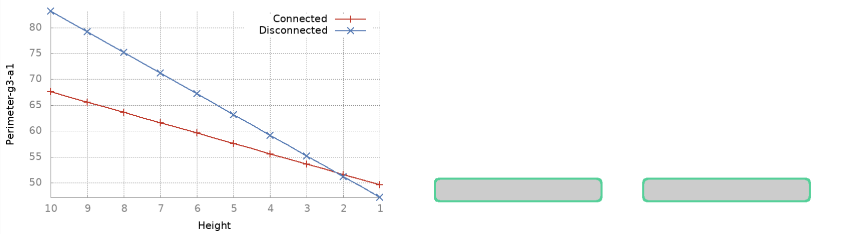

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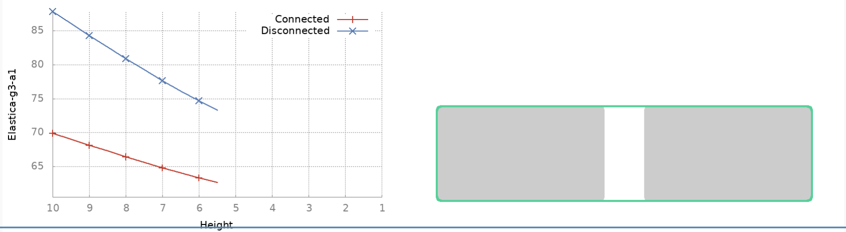

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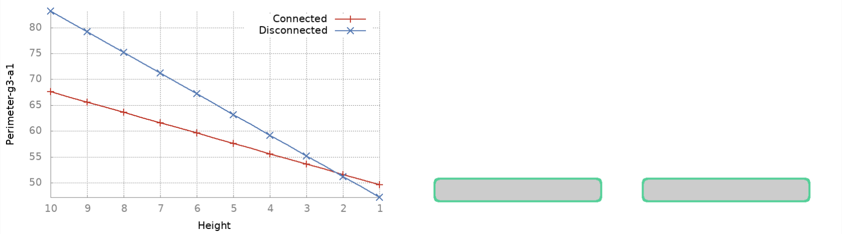

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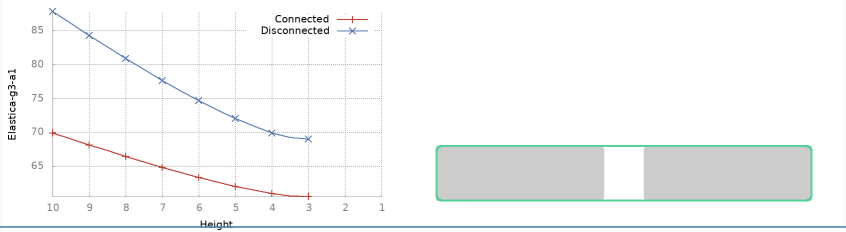

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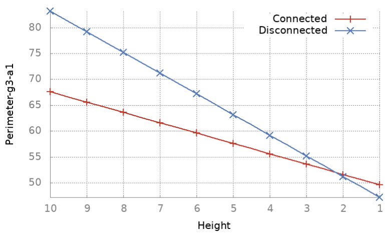

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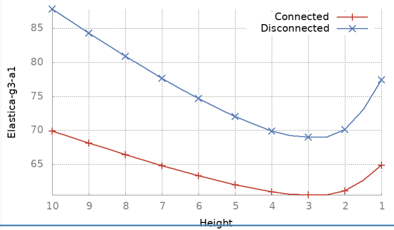

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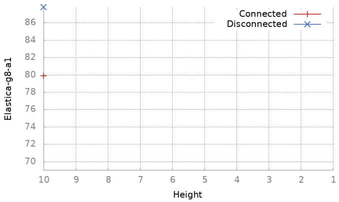

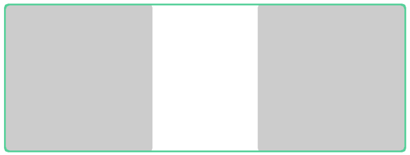

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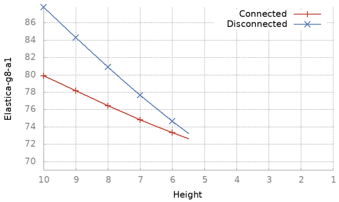

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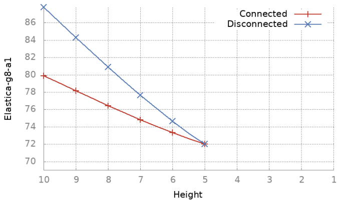

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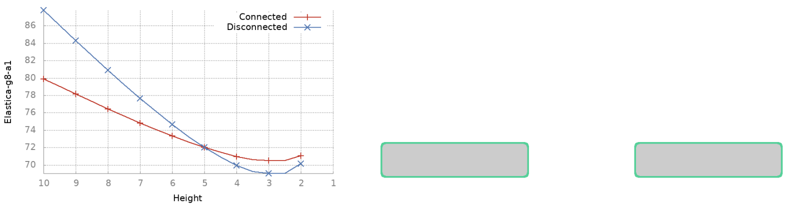

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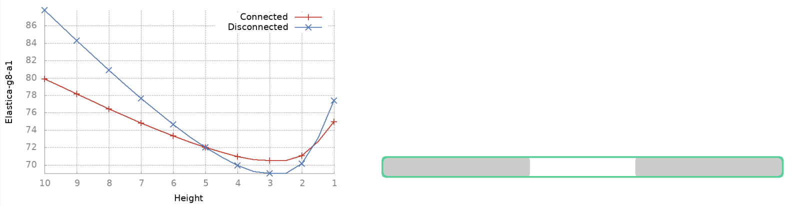

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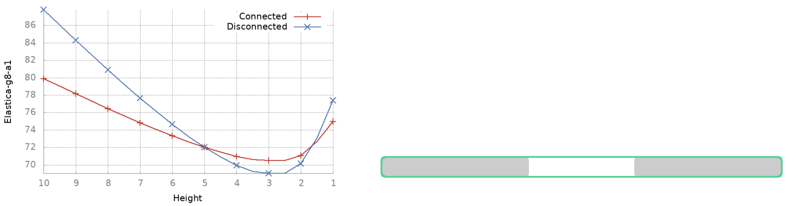

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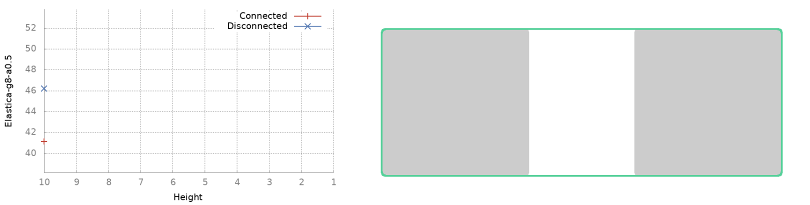

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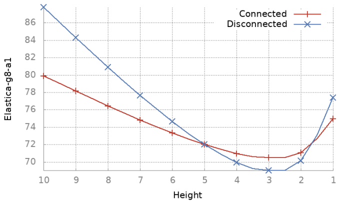

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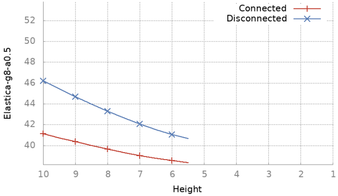

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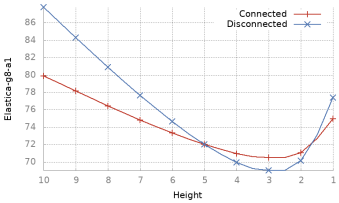

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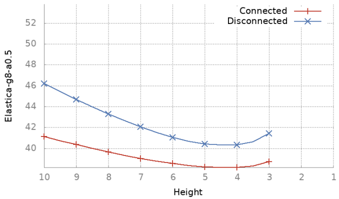

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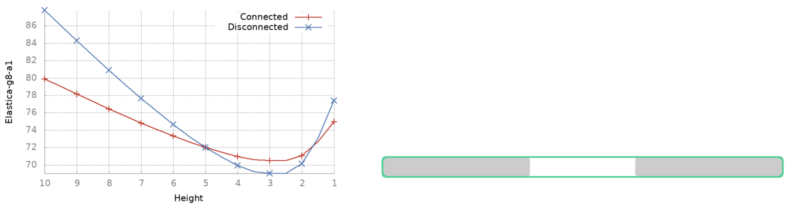

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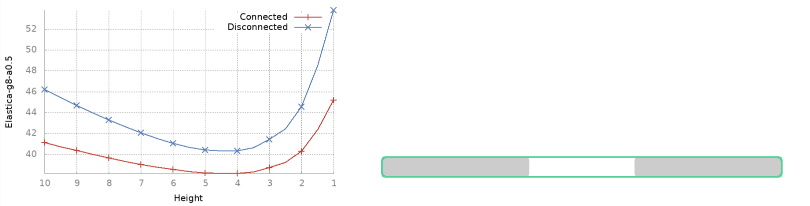

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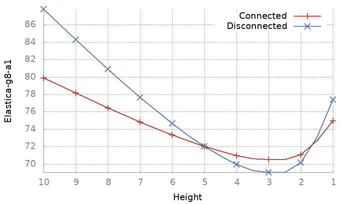
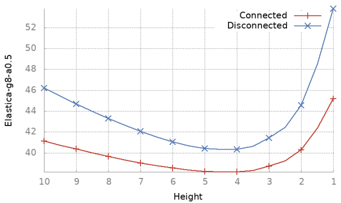


$$\min_{X \in \Omega} \int_{\partial X} \alpha + \beta \kappa^2 ds. \quad - \quad \text{The elastica energy}$$



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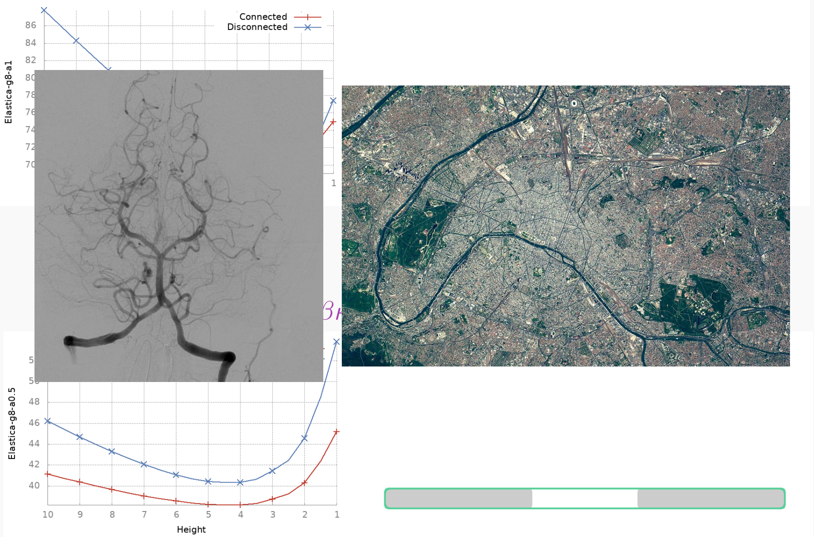

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State-of-the-art

Continuous setting: Define the energy over the whole domain and minimize the elastica with respect the level-curves (Chan, S. H. Kang, Kang, and Shen 2002).

$$\int_{\Omega} \left(\alpha + \beta \nabla \cdot \left(\frac{\nabla f_I}{\|\nabla f_I\|} \right)^2 \right) \|\nabla f_I\| d\Omega.$$

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T-junctions matching
Masnou and Morel 1998

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Triple cliques Nieuwenhuis, Toeppe, Gorelick, Veksler, and Boykov 2014	Global formulation, quadratic non-submodular energy. Limited precision.

Motivation

Goals

Models based on the minimization of the elastica energy

	Continuous	Discrete	**Digital**
Numerical instability	Yes	No	No
Suitable for digital sets	No	No	Yes
Rounding issues	Yes	No	No
Contour completion	Partial	Partial	Extended
Global optimum (Free elastica)	-	-	Yes

Outline

1. Motivation
   - Image analysis and geometric priors
   - Elastica model and completion property
   - State-of-the-art
2. Contribution
   - Digital sets and convergent estimators
   - Elastica minimization via graph-cuts
3. Conclusion and perspectives

Digital sets and convergent estimators

- Digital grid particularities and restrictions.
- Multigrid convergence of geometric estimators.

Digital sets and convergent estimators

Digital set peculiarities

Where can we do better?

- Most of models neglect the digital character of digital images and ignore the fact that geometric measurements (mainly those local as tangent and curvature) in such objects should be done with a definition of *convergence* that is specific for digital shapes.

Digital sets and convergent estimators

Digital set peculiarities

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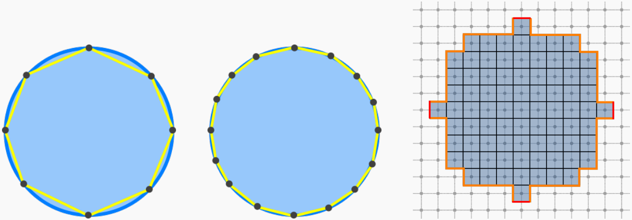
Exact sampling x digitization



Digital sets and convergent estimators

Digital set peculiarities

Where can we do better?

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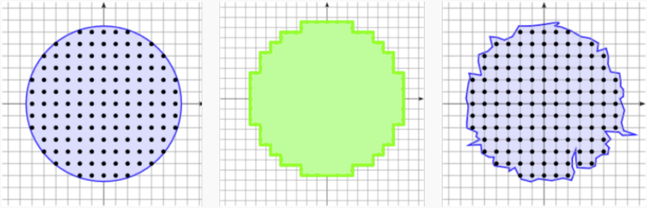
Digitization ambiguity



Digital sets and convergent estimators

Multigrid convergent estimators

Definition (Multigrid convergence)

Let $\mathcal{X}$ be a family of shapes in $\mathbb{R}^n$ and u a geometric quantity that is defined for every shape $X \in \mathcal{X}$. Further, let $D_h(X)$ denote the digitization of X with grid step h.

The estimator $\hat{u}$ is multigrid convergent for $\mathcal{X}$ if and only if, for any $X \in \mathcal{X}$ there exists $h_X > 0$ such that for every $0 < h < h_X$

$$|\hat{u}(D_h(X)) - u(X)| \leq \tau(h), \quad \text{with } \lim_{h \to 0} \tau(h) = 0.$$

Digital sets and convergent estimators

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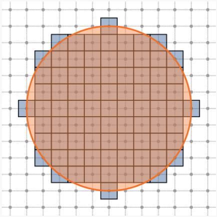
Multigrid convergent estimator of area

$$\widehat{Area(X)} = h^2 |D_h(X)|.$$

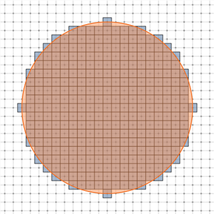
Motivation

Multigrid convergent estimators

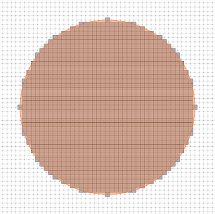
Disk of radius $5 (Area \approx 78.54)$.



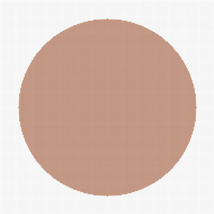
$h = 1.0,\ \widehat{A} = 81.$



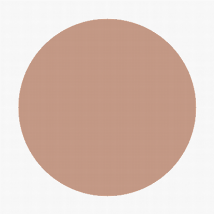
$h = \frac{1}{2},\ \hat{A} = 79.25.$



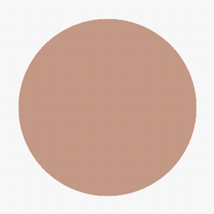
$h = \frac{1}{4},\ \hat{A} = 78.56.$



$h = \frac{1}{16},\ \hat{A} = 78.44.$



$h = \frac{1}{32},\ \hat{A} = 78.5.$

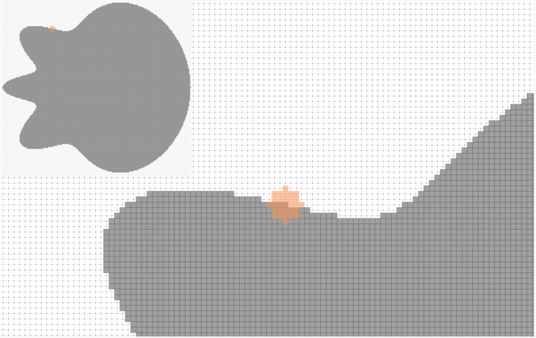


$h = \frac{1}{64},\ \hat{A} = 78.53.$

Digital sets and convergent estimators

Multigrid convergent estimators

- Integral Invariant (II) Coeurjolly, Lachaud, and Levallois 2013
  - Proved multigrid convergent for C^2 convex shapes with bounded curvature.



$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$

Digital sets and convergent estimators

Conclusion

- Digital sets are ambiguous and are constrained to the digital grid.
- The multigrid convergence is an adapted definition of convergence for geometric estimation on digital sets.

Digital sets and convergent estimators

Conclusion

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- The multigrid convergence is an adapted definition of convergence for geometric estimation on digital sets.

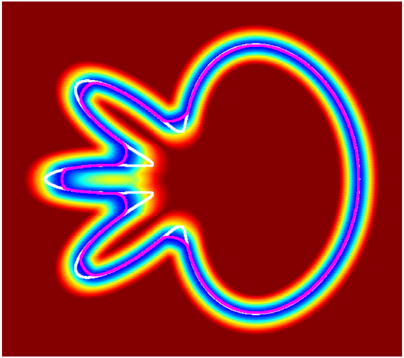
Can we construct optimization models using multigrid convergent estimators?

Elastica minimization via graph-cuts

- Balance coefficient to estabilize curvature estimation.
- Set up a graph whose minimum cut approximates the zero level set of the balance coefficient.
- GraphFlow algorithm. Up to 10x faster than FlipFlow.

Non-submodular elastica

Balance coefficient



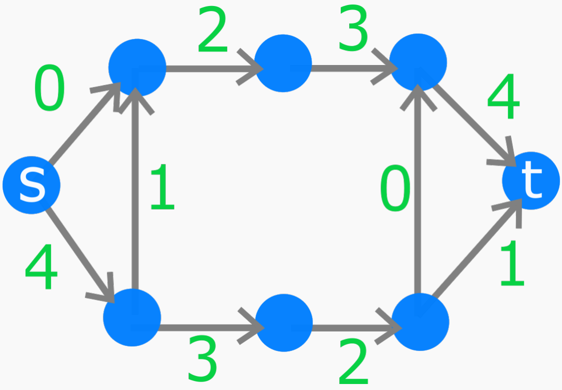
- Balance coefficient

$$u_r(D,p) = \left(\frac{\pi r^2}{2} - |B_r(p) \cap D|\right)^2$$

- White contour: contour of the shape
- Pink contour: ϵ-level set of the balance coefficient

Elastica minimization via graph-cuts

Graph cut



Elastica minimization via graph-cuts

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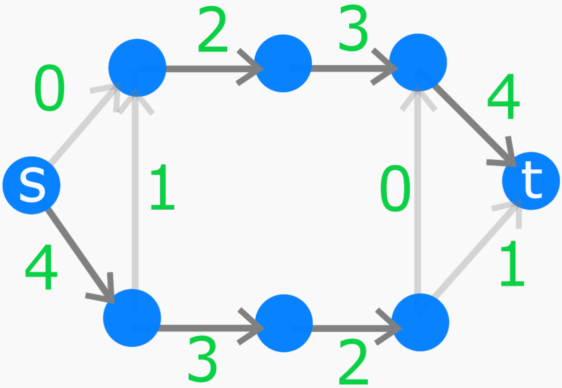

Elastica minimization via graph-cuts

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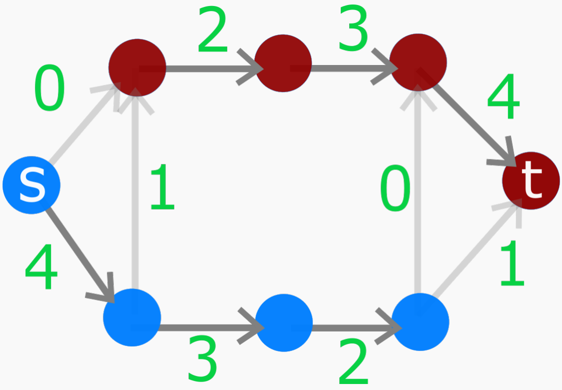
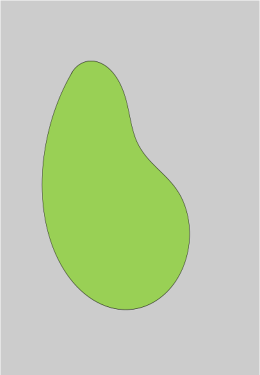


Elastica minimization via graph-cuts

Building the graph

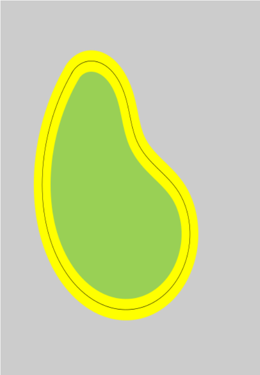


Elastica minimization via graph-cuts

Building the graph

- Optimization band

$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$



Elastica minimization via graph-cuts

Building the graph

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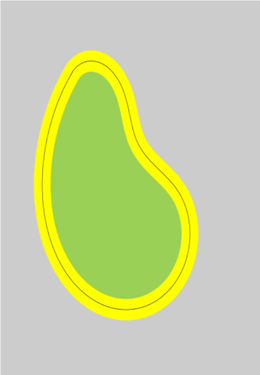
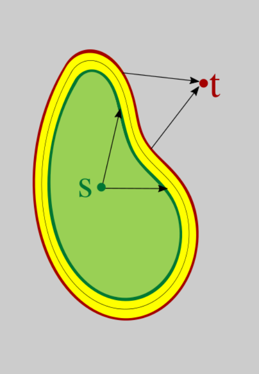
$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$
$$F(D) := D \setminus O(D)$$



Elastica minimization via graph-cuts

Building the graph



- Optimization band

$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$
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- Graph $\mathcal{G}_D(\mathcal{V}, \mathcal{E}, c)$

$$\mathcal{V} = \{v_p \mid p \in O(D)\} \cup \{s, t\}$$
$$\mathcal{E} = \{\{v_p, v_q\} \mid p, q \in O(D) \text{ and } q \in \mathcal{N}_4(p)\} \cup \mathcal{E}_{st}$$
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Elastica minimization via graph-cuts

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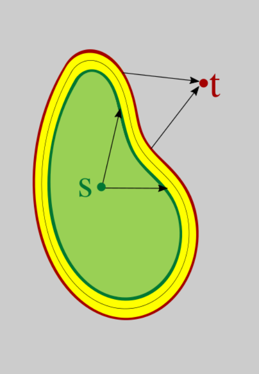

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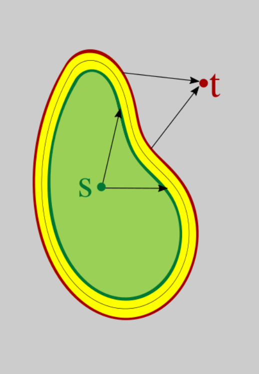

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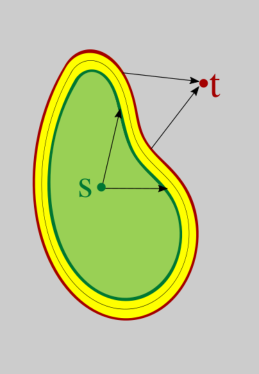

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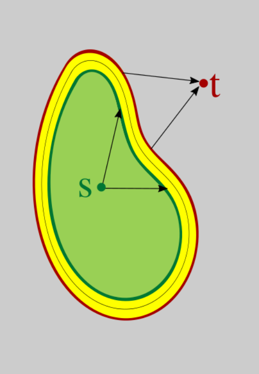
- Edge's weight

edge e	$\mathbf{c(e)}$
$\{v_p, v_q\}$	$\frac{1}{2}\left(u_r(D,p) + u_r(D,q)\right)$
(s, v_p)	M
(v_p, t)	M

Elastica minimization via graph-cuts

Building the graph



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(v_p, t)	M

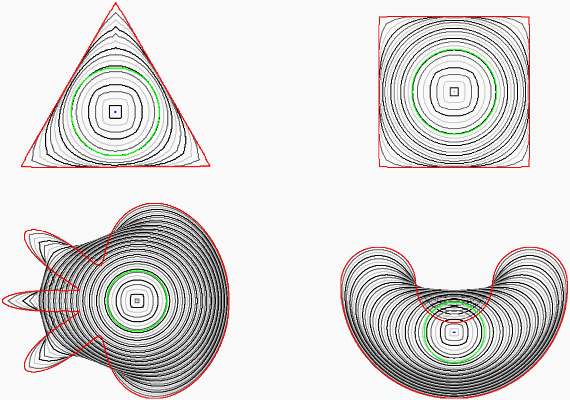
- Digital shape update

$$D^{(k+1)} = F(D^{(k)}) + S^{(k)}$$

Elastica minimization via graph-cuts

Shape evolution

$$\alpha = 1/8^2, \beta = 1.$$



Elastica minimization via graph-cuts

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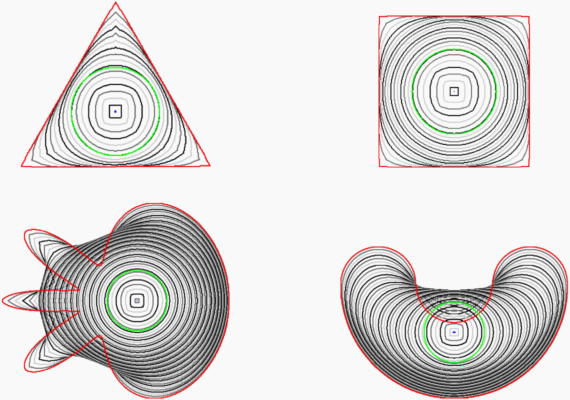
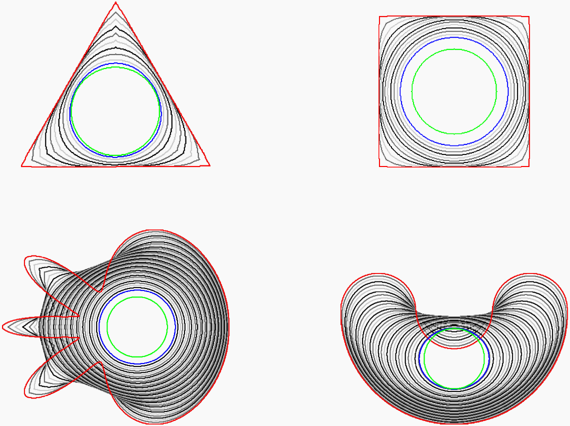


- What if we stop the evolution when elastica increases?

Elastica minimization via graph-cuts

Shape evolution

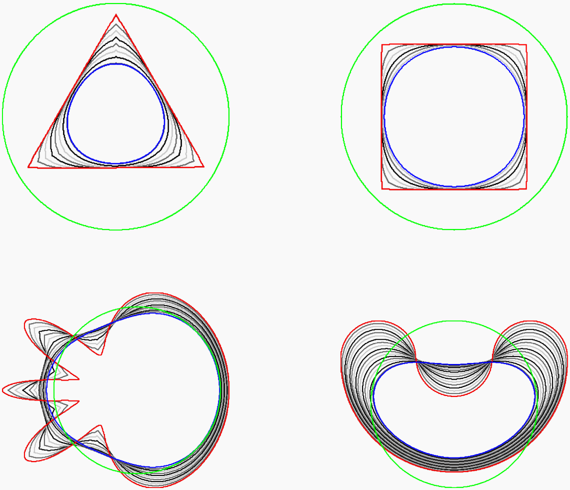
Stop if elastica increases $(\alpha = 1/8^2, \beta = 1)$



Elastica minimization via graph-cuts

Shape evolution

Stop if elastica increases $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

The a-probe set

Definition (a-probe set)

Let $D \subset \Omega \subset \mathbb{Z}^2$ a digital set and a a natural number. The a-probe set of D is defined as

$$\mathcal{P}_a(D) = D \cup \bigcup_{a' < a} D^{+a'} \cup D^{-a'},$$

where $D^{+a}(D^{-a})$ denotes a dilation(erosion) by a disk of radius a.

Candidate selection

$$sol(D^{(k)}) \longleftarrow \bigcup_{D' \in \mathcal{P}_a(D^{(k)})} \left\{ F^{(k)} + S \mid mincut(S, \mathcal{G}_{D'}) \right\}$$

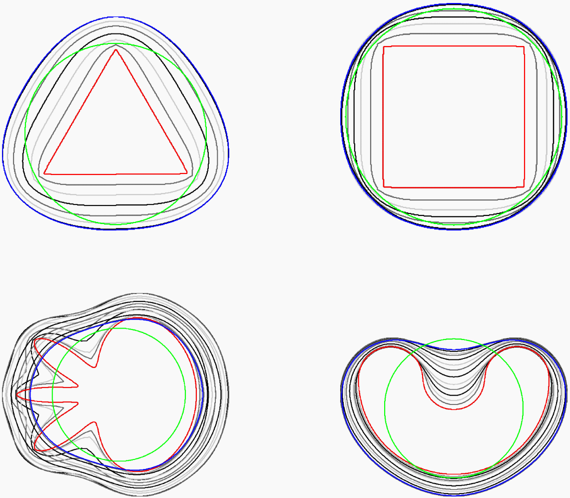
Candidate validation

$$D^{(k+1)} \longleftarrow \underset{D' \in sol(D^{(k)})}{\arg\min} \; \hat{E}_{\boldsymbol{\theta}}(D')$$

Elastica minimization via graph-cuts

Shape evolution with α-probe set

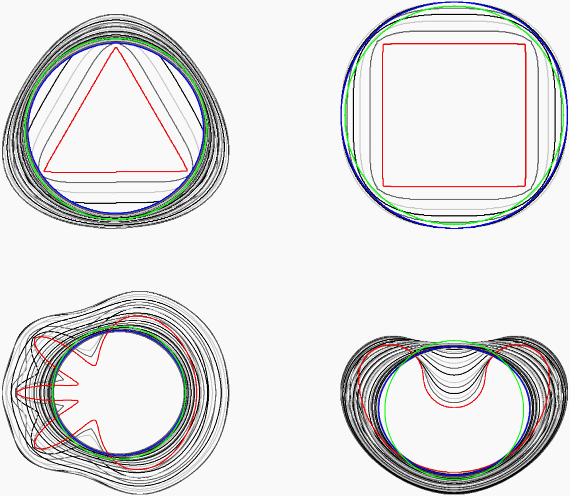
Stop if elastica increases $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

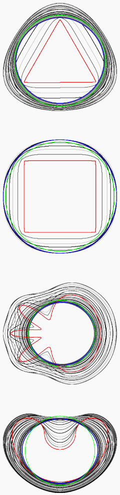
Shape evolution with α-probe set

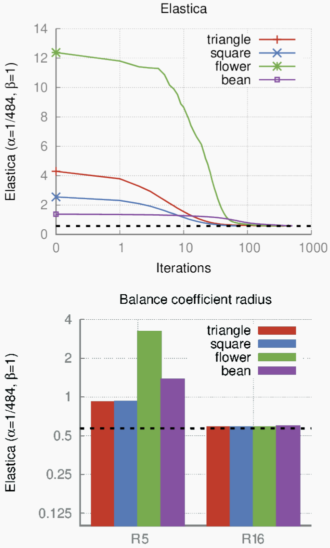
Always update $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

Shape evolution with α-probe set





Elastica minimization via graph-cuts

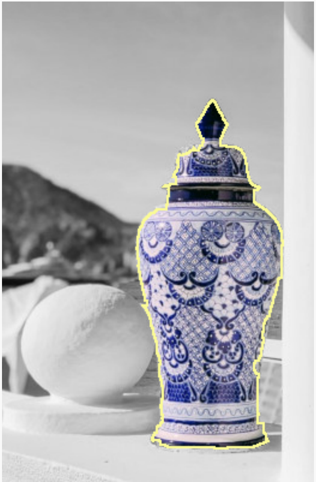
Contour correction

Initial segmentation

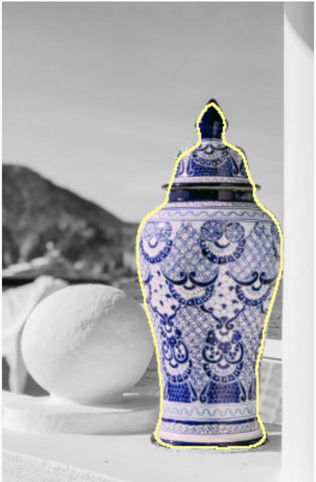
$0.825s$ (3 it)

Elastica minimization via graph-cuts

Contour correction



Initial segmentation



$0.746s$ (3 it)

Elastica minimization via graph-cuts

Contour correction

Initial segmentation

$1.1s$ (3 it)

Elastica minimization via graph-cuts

Contour correction

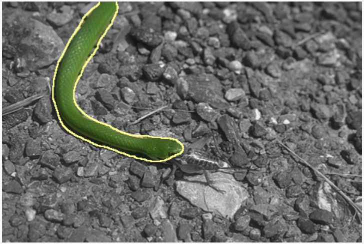
Initial segmentation

$10s$ (30 it)

Elastica minimization via graph-cuts

Contour completion

Initial segmentation



$17s$ (62 it)

Conclusion

Summary of models

Model	Implementation	Running time	Free elastica	Constrained elastica	Image term
LocalSearch	medium	slow	yes(**opt**)	yes	no
FlipFlow	hard	acceptable	yes	no	yes
(BalanceFlow)	medium	acceptable	yes	no	yes
GraphFlow	easy	fast	yes(**opt**)	no	yes

Table: **Models summary.** The qualitative attributes are relative, e.g., the GraphFlow presents the lowest running time while LocalSearch presents the highest.

Conclusion

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Table: Models summary. The qualitative attributes are relative, e.g., the GraphFlow presents the lowest running time while LocalSearch presents the highest.

	Pixels	LocalSearch	FlipFlow	BalanceFlow	GraphFlow
Triangle	8315	4.8s/it	0.4s/it	0.38s/it	0.14s/it
Square	12769	2s/it	0.51s/it	0.47s/it	0.12s/it
Ellipse	10038	3.1s/it	0.64s/it	0.57s/it	0.1s/it
Flower	26321	12.3s/it	1.23s/it	0.94s/it	0.14s/it
Bean	25130	6.4s/it	1.2s/it	1.17s/it	0.16s/it

Table: Free elastica running times. Running time and input size for the free elastica experiment.

Conclusion

Summary of models

- We proposed a digital elastica optimization model.
- GraphFlow is extendable (suitable for data terms) and our fastest model.
- Contour completion is achieved in some cases.

Pros

- Topology is flexible.
- Easily parallelizable.
- Neighborhood flexibility.

Cons

- Susceptible to bad local minimum (we can ameliorate with a better definition of the neighborhood).

Conclusion
Perspectives

- **GraphFlow and perimeter**: enrich the cost function of GraphFlow with the weights defined in Boykov and Kolmogorov 2003.
- **Different neighborhoods**: random, linear extension.
- **Dynamic radius**: use the parameter free Maximal Digital Circular Arcs estimator of curvature to adapt the estimation disk radius to use.
- **Multiresolution**: Improve running time; or improve estimator precision.
- **Image analysis applications**: Objective comparison of our method and competitive ones (e.g. study quantitative measurements such as the ratio of inflexion points for the contour correction application) .
- **Global formulation and multigrid convergent estimators**: Does a practical model for elastica exist?

Thank you!

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