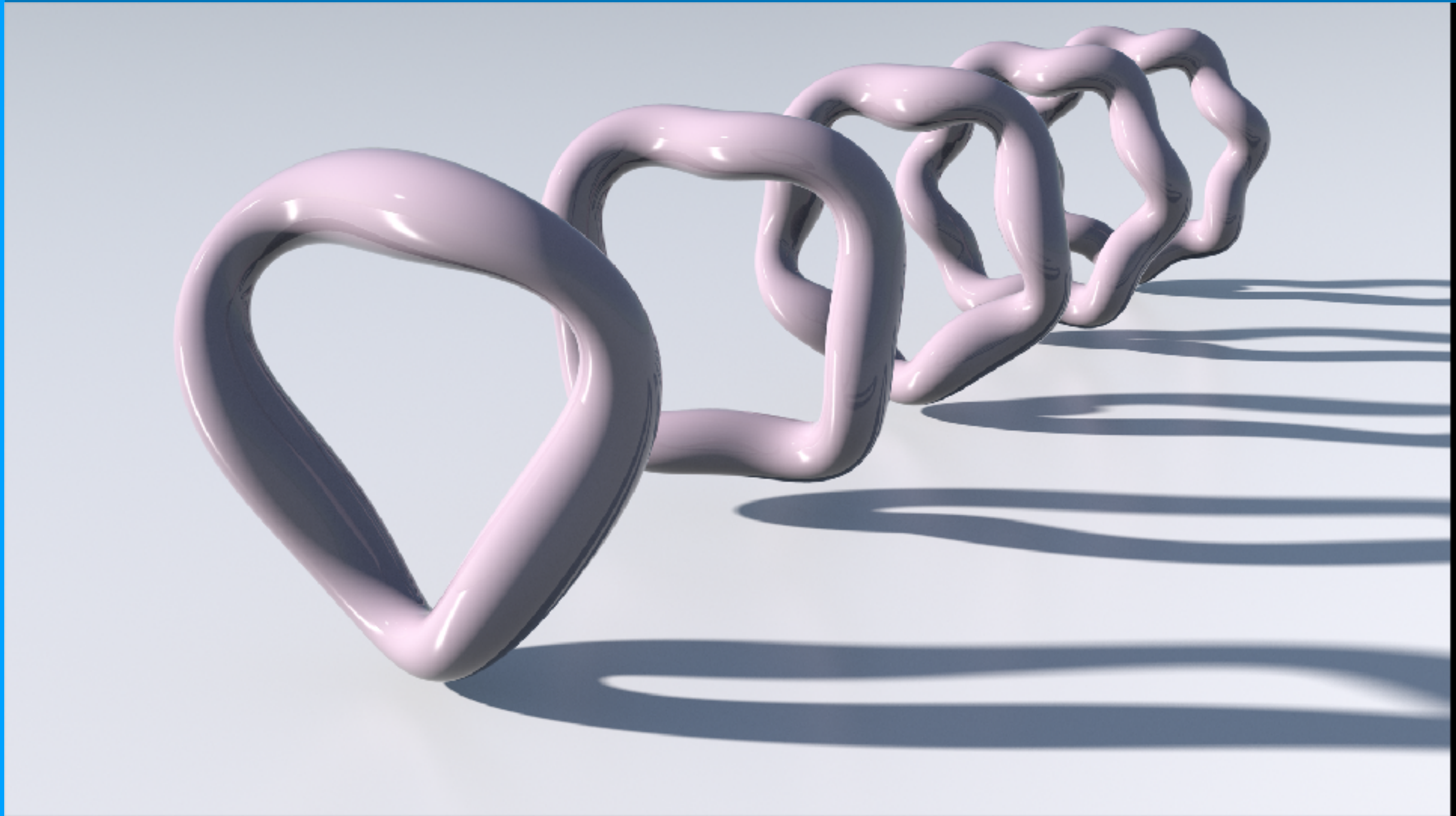
TU Berlin

Computing Constrained Willmore Surfaces



Ulrich Pinkall

Yousuf Soliman

Felix Knöppel

Olga Diamanti

Albert Chern

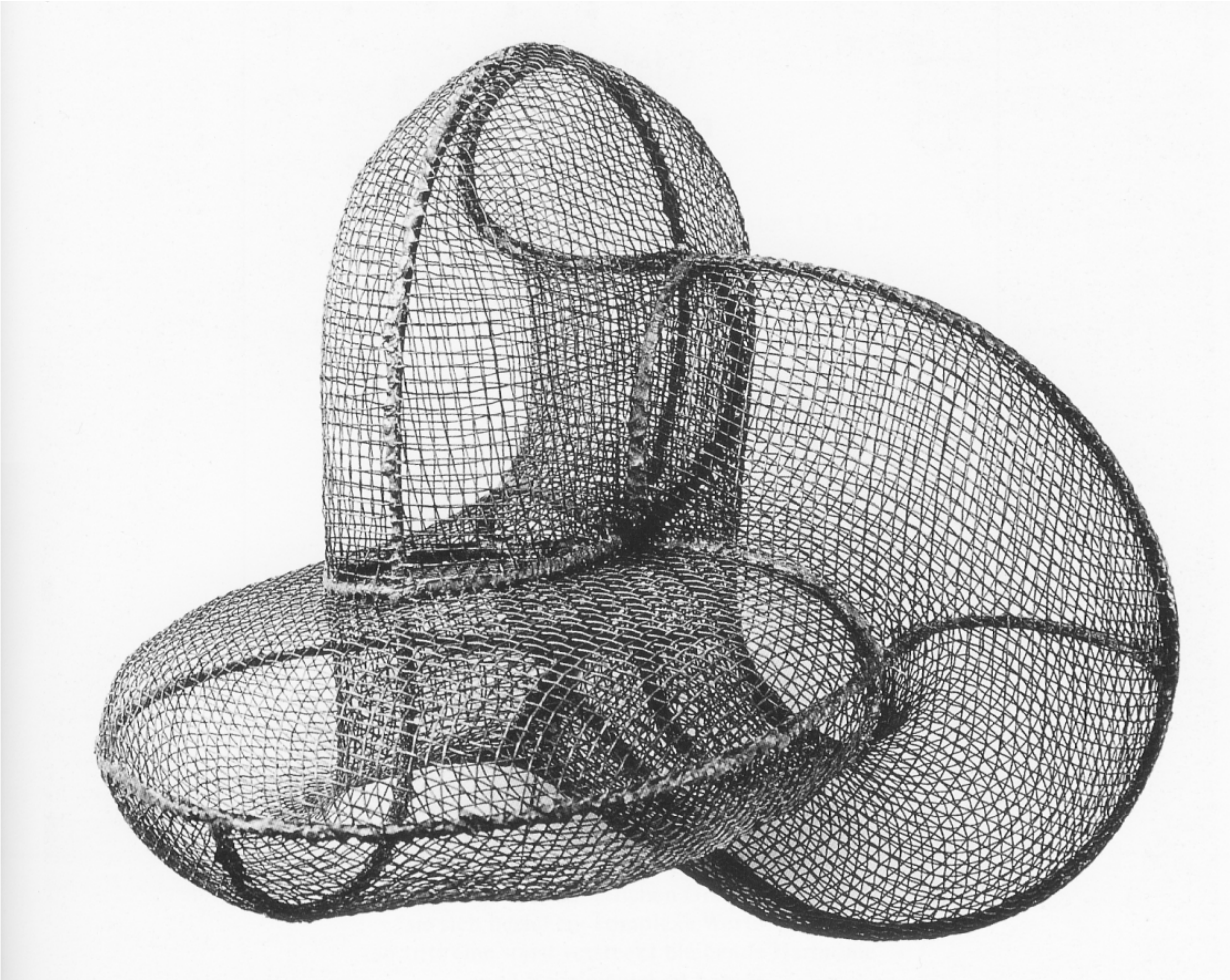
Peter Schröder

CIRM March 2021

Werner Boy
1903:

First immersion

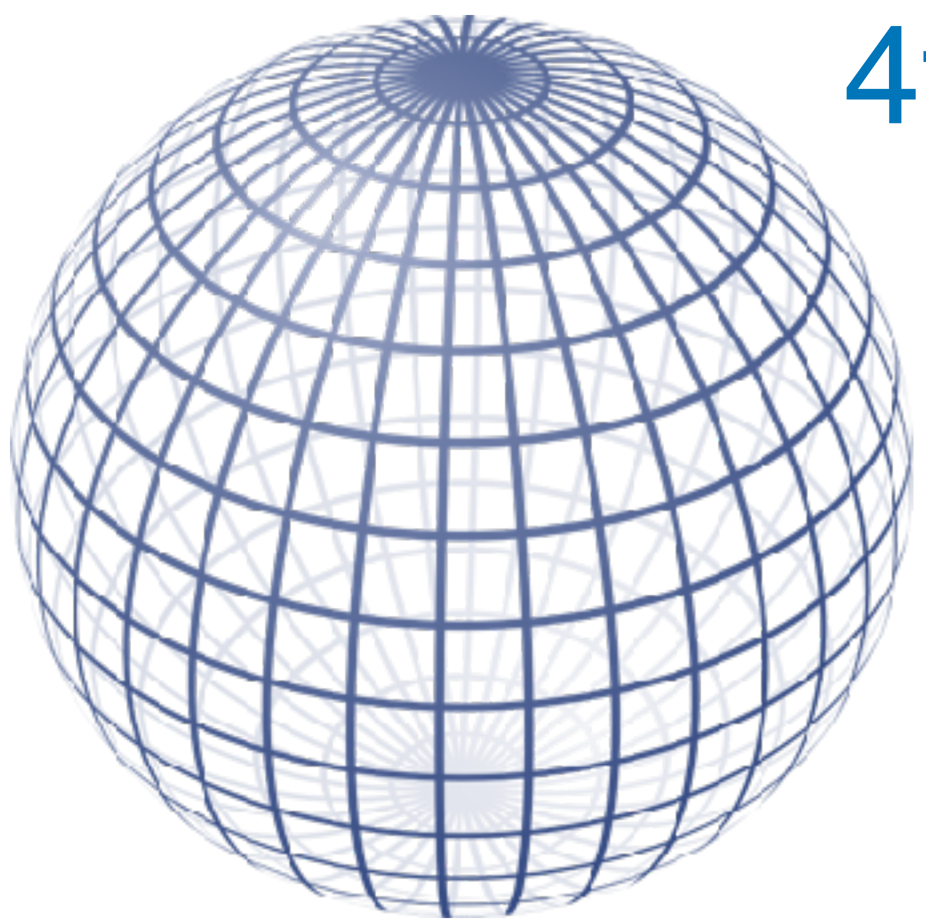
$$f : \mathbb{RP}^2 \to \mathbb{R}^3$$



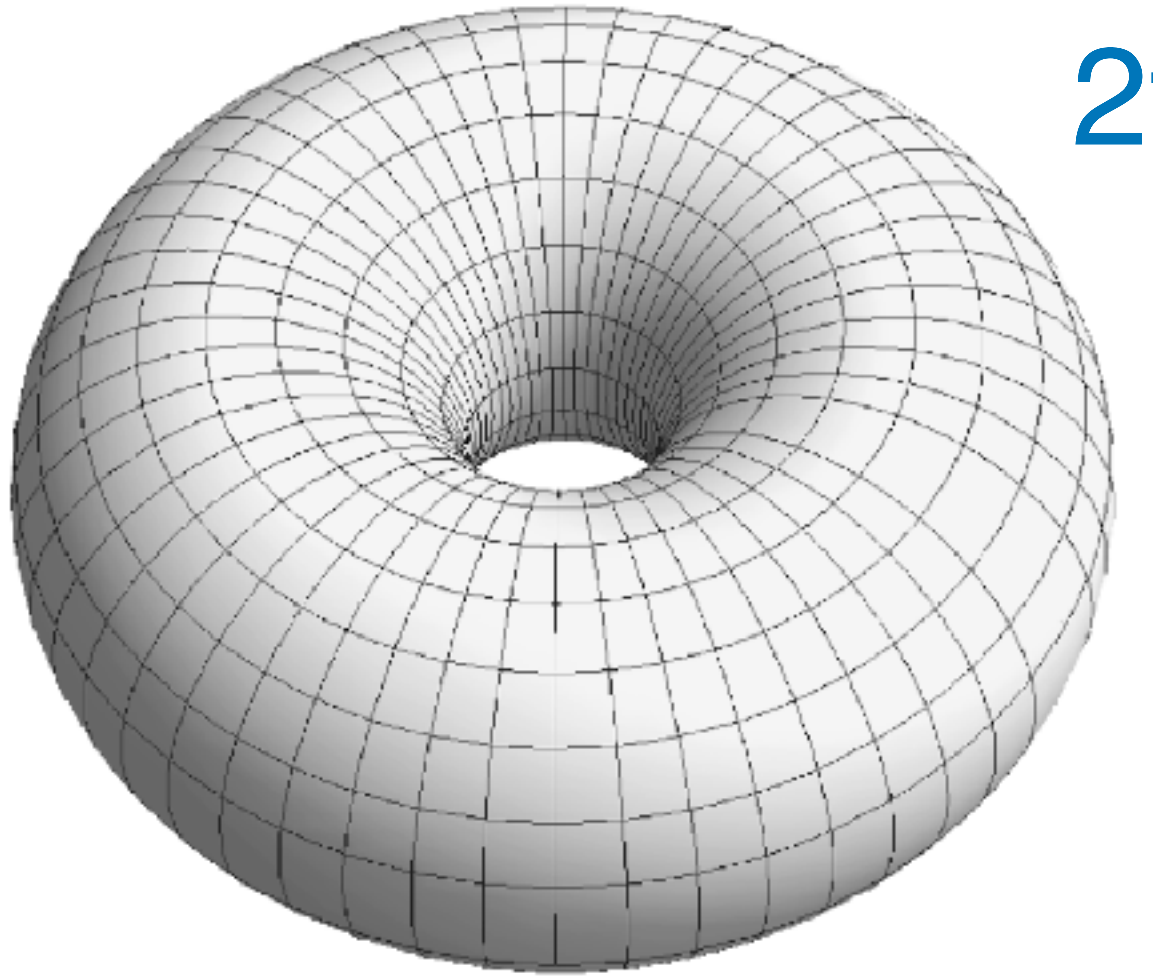
$$f : M \to \mathbb{R}^3$$

is called a **Willmore surface** if it is a critical point of

$$W(f) = \int_M H^2$$



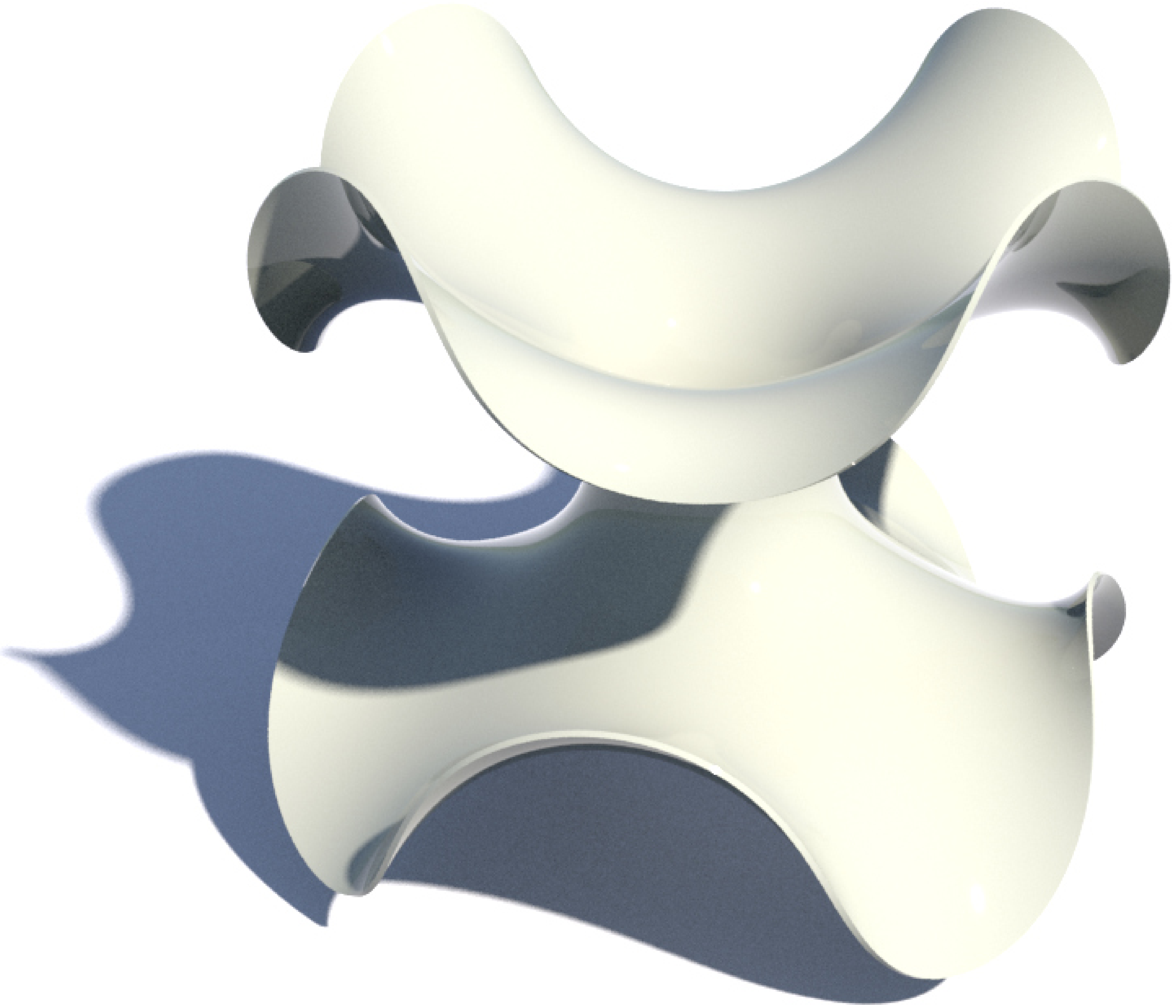
4π



2π²

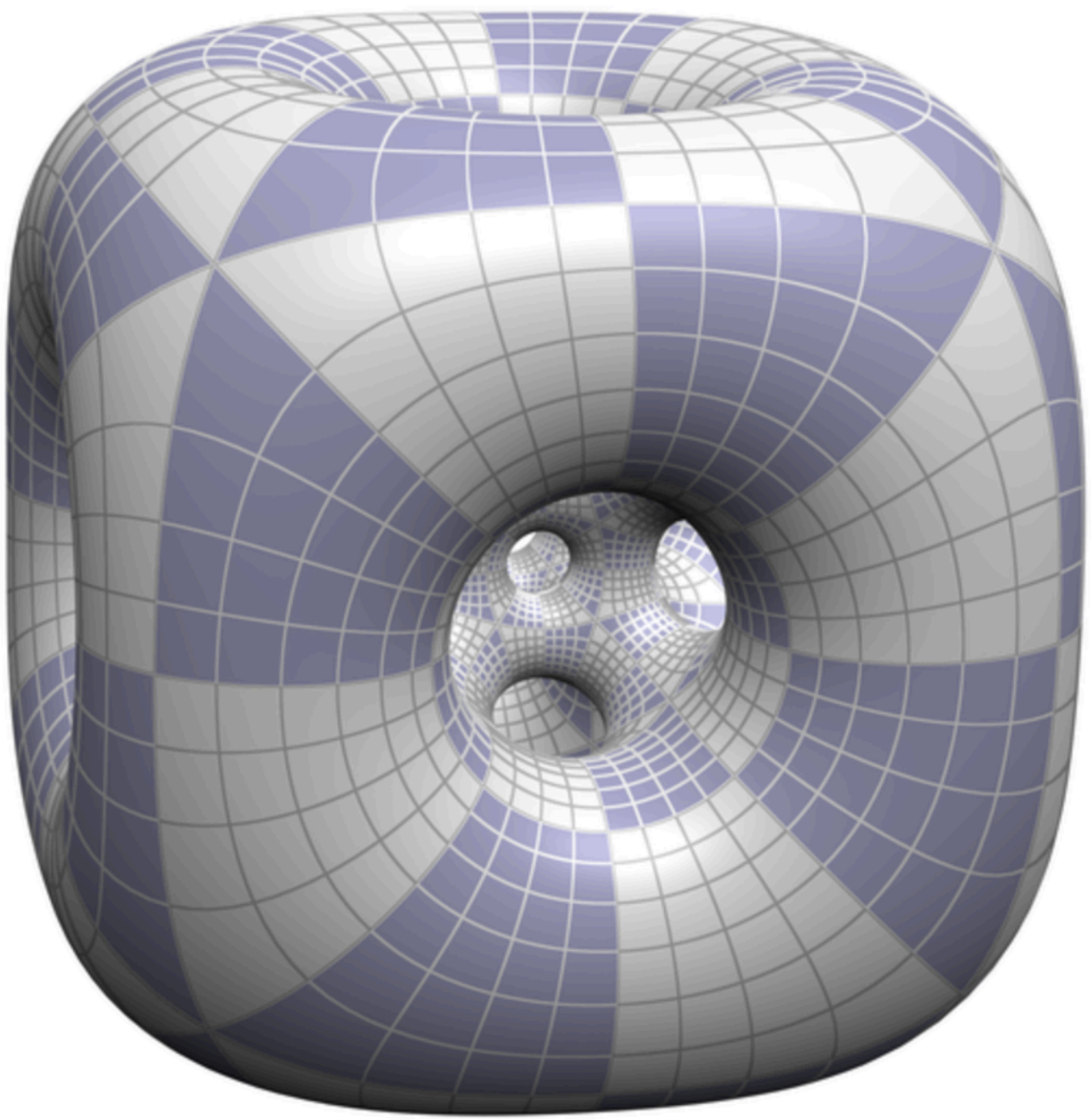
12π

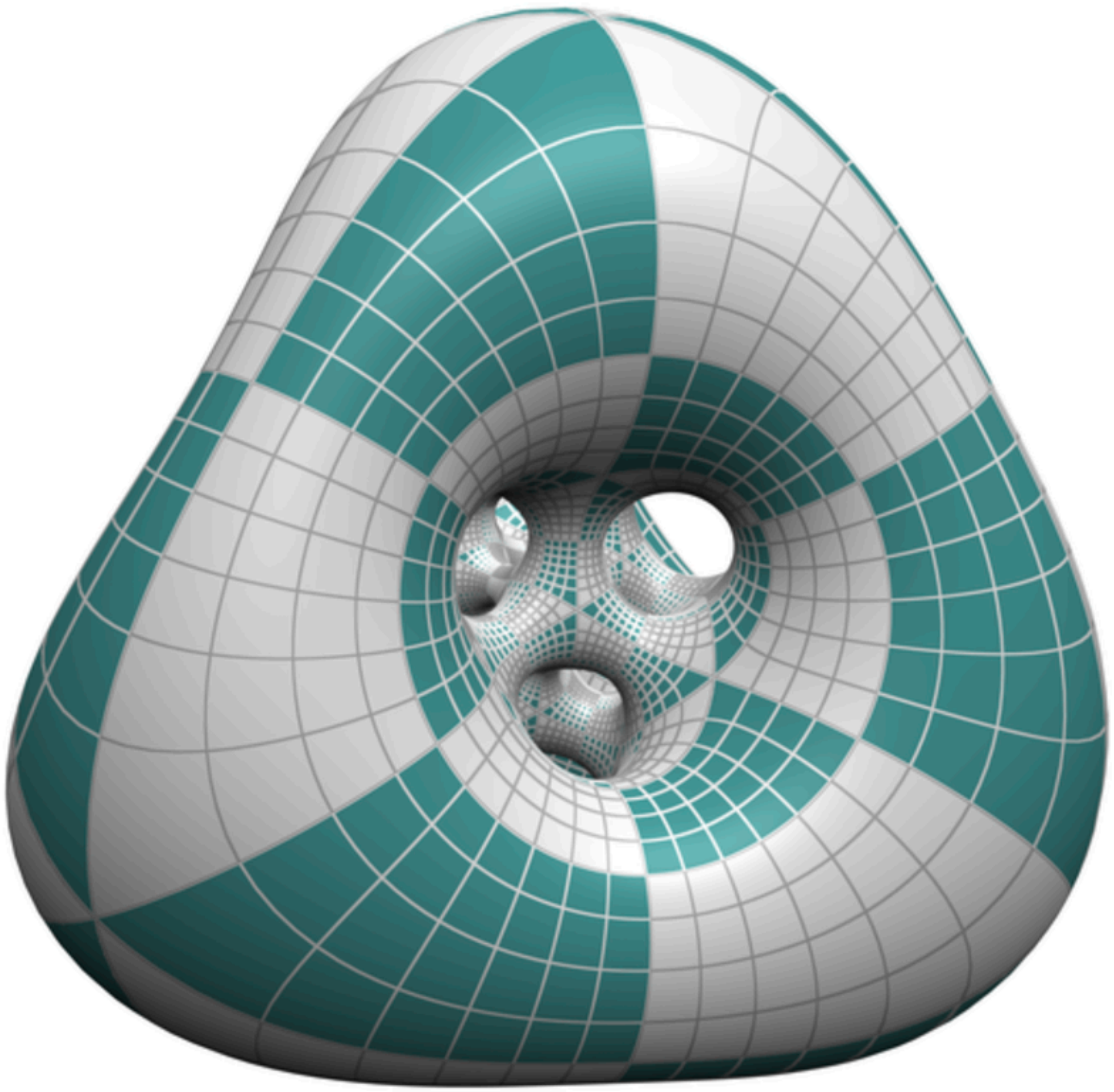
Minimal surfaces in $\mathbb{R}^3$



Images by Oliver Gross

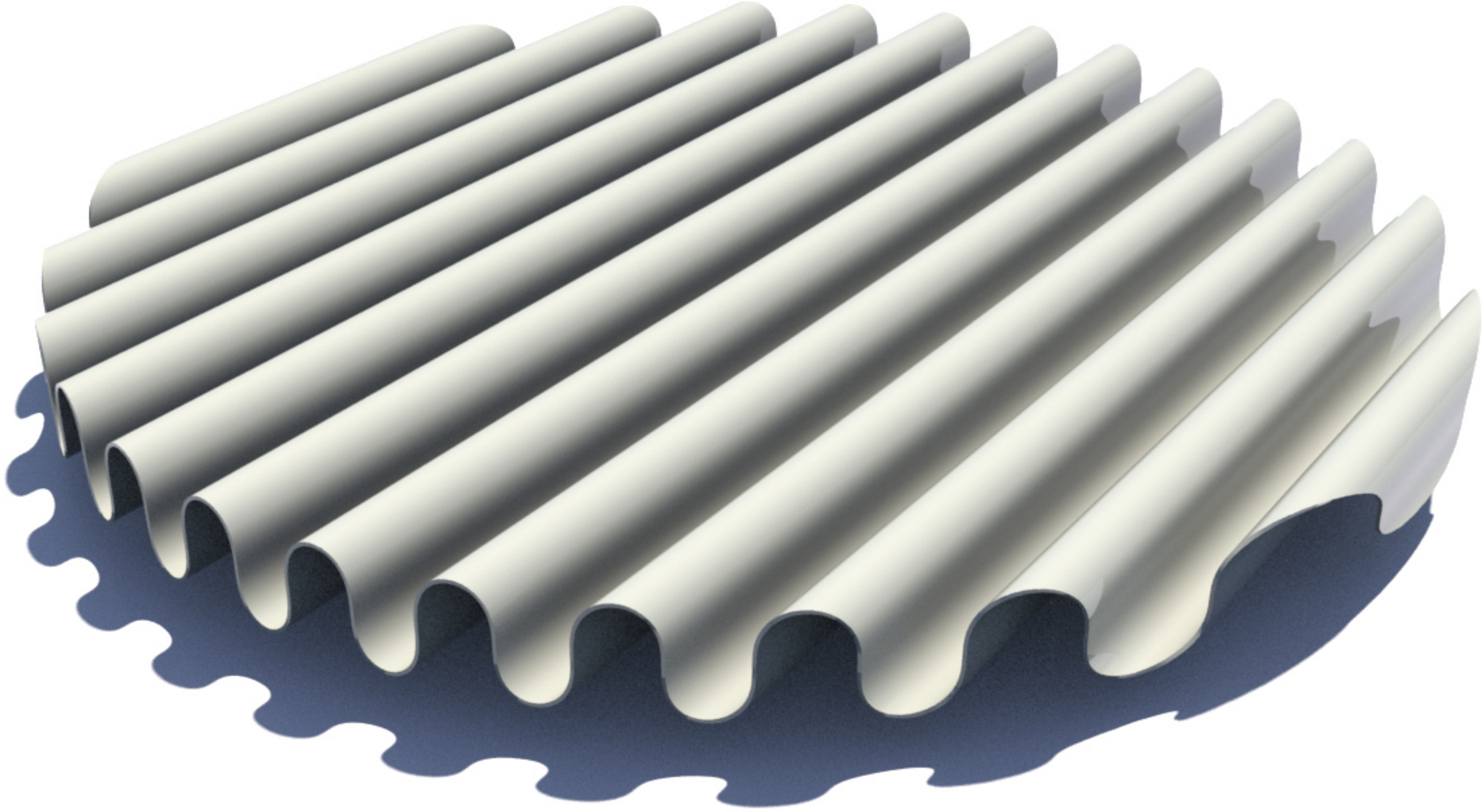
Minimal surfaces in S^3

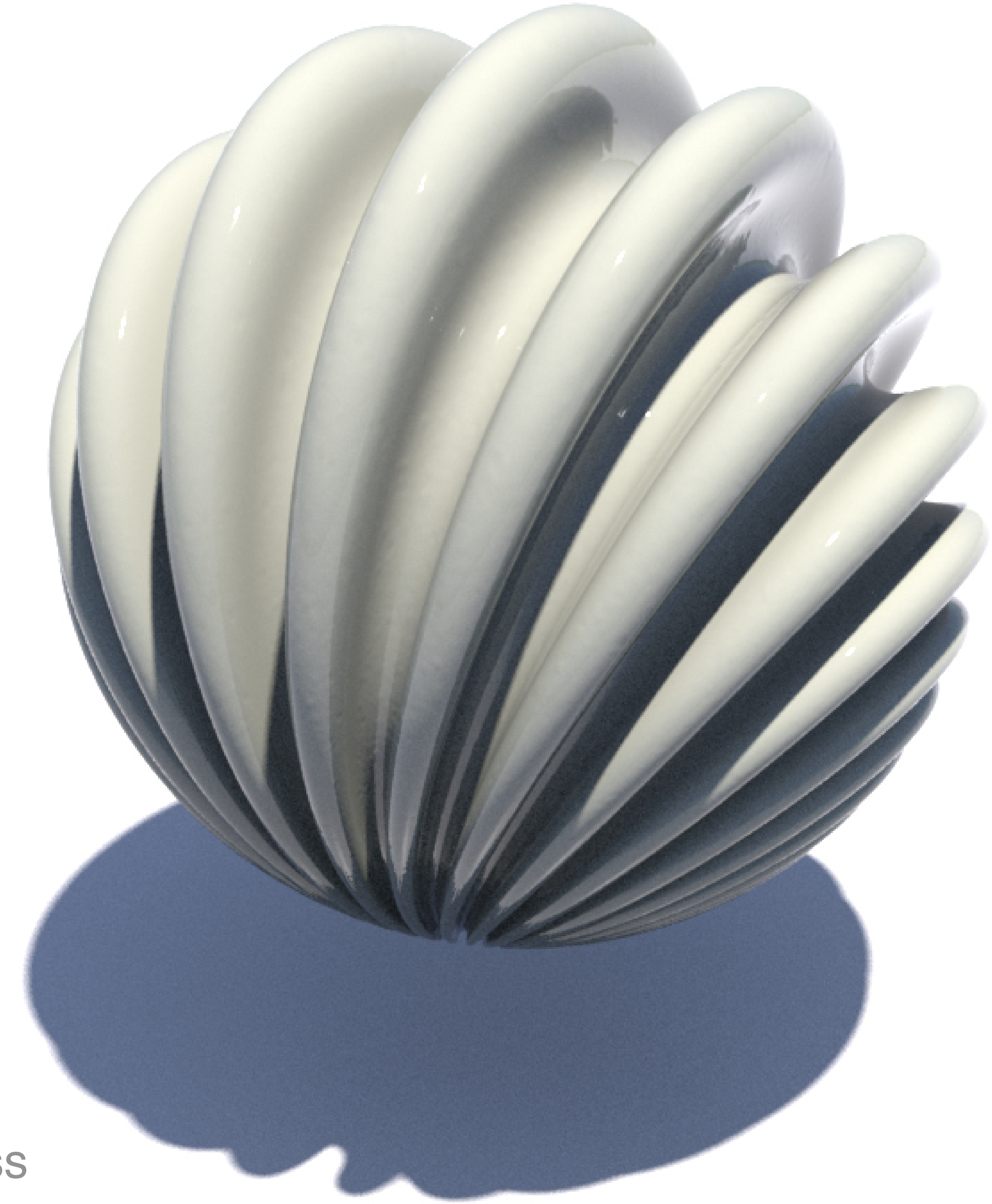




Images by Nicholas Schmitt

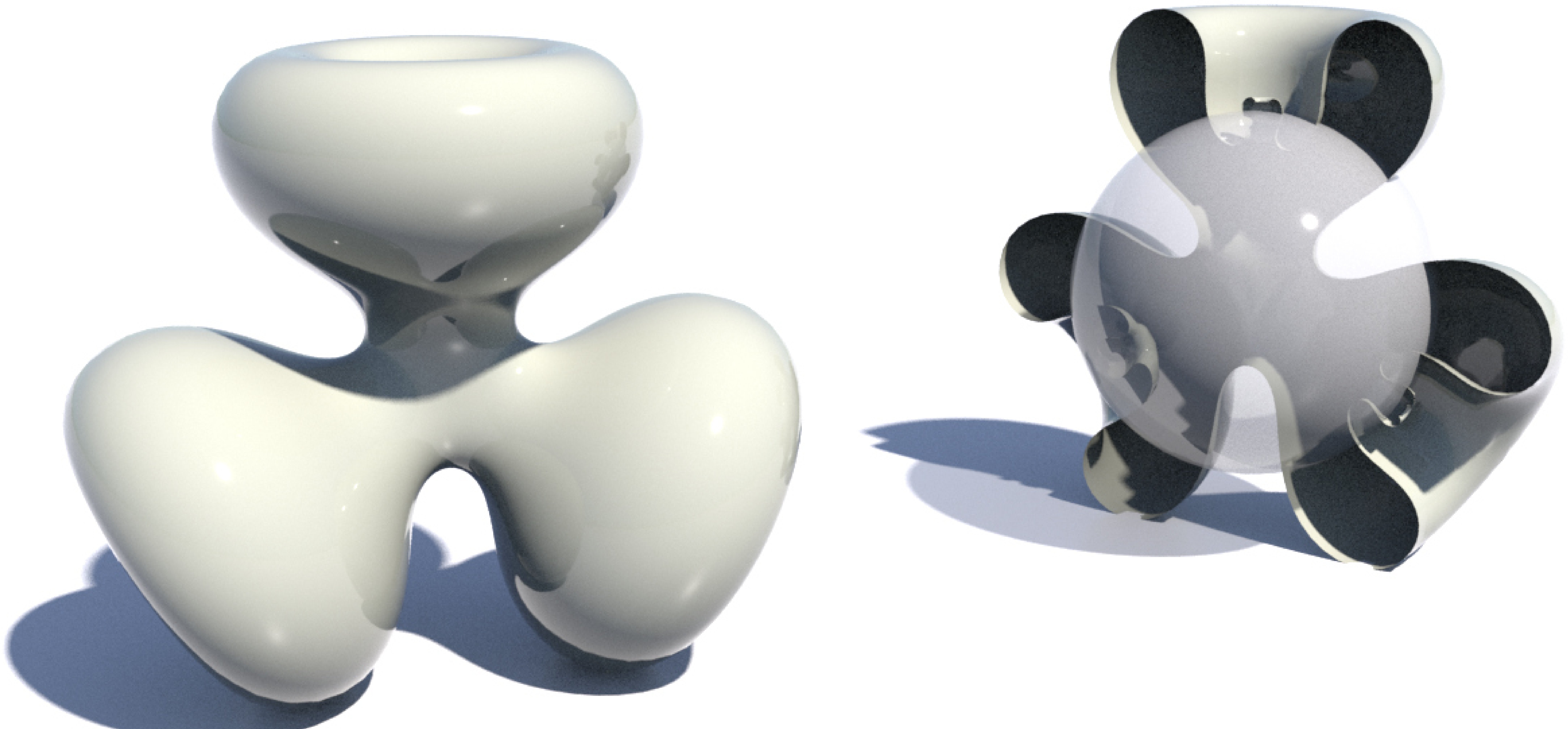
Minimal surfaces in H^3





Images by Oliver Gross

Minimal surfaces in H^3



Images by Oliver Gross

A Riemann surface is defined as a 2-dimensional manifold M together with an endomorphism field $J \in \Gamma\mathrm{End}(TM)$ with

$$J^2 = -I$$

A Riemann surface has a canonical orientation and a canonical conformal structure, comprising those Riemannian metrics $\langle \, , \rangle$ that satisfy

$$\langle JX, JY \rangle = \langle X, Y \rangle$$


M
JX
X
Surface by Nicholas Schmitt

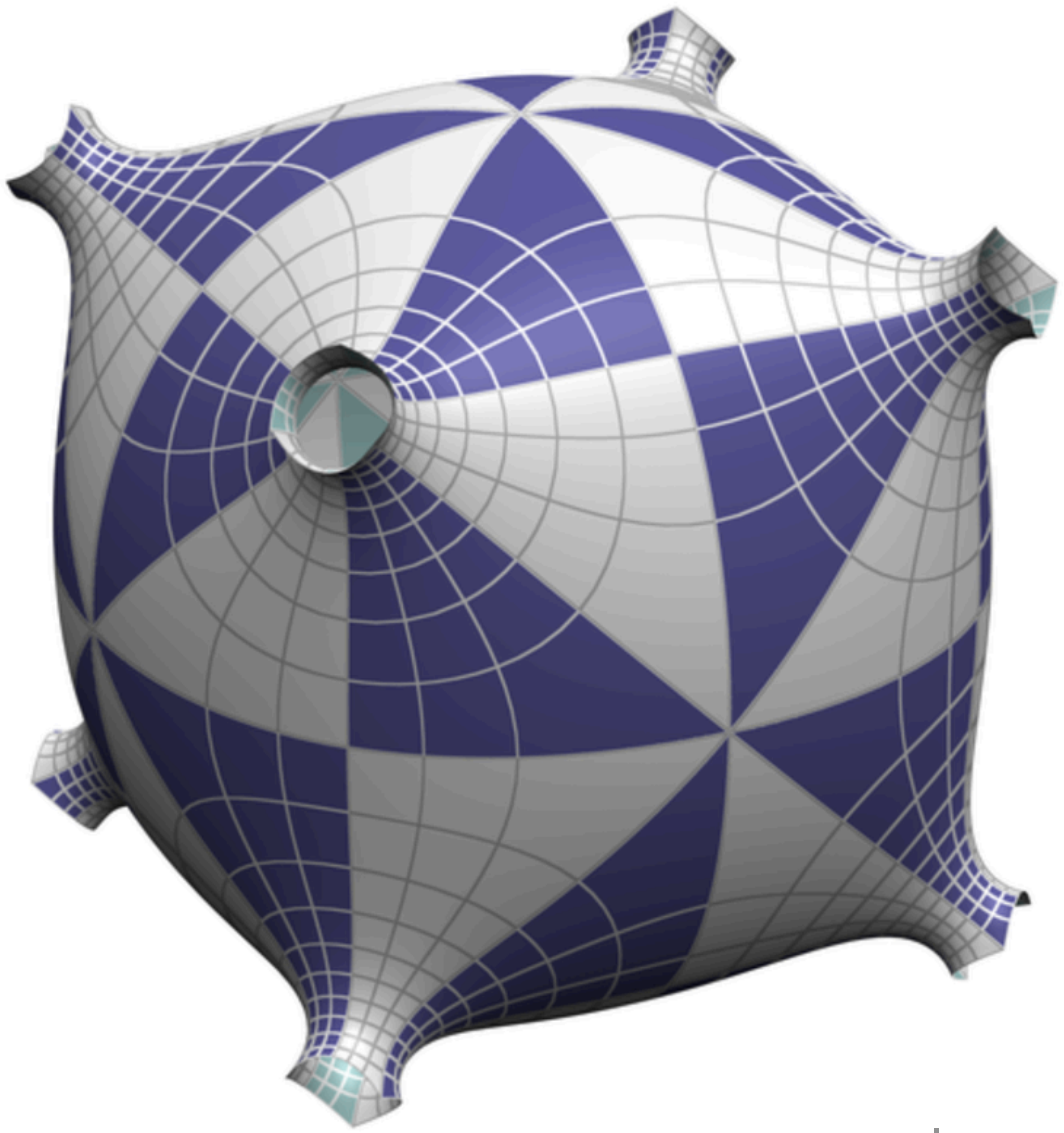
Theorem (Garsia 1961): Every compact Riemann surface (M, J) admits a conformal immersion $f : M \to \mathbb{R}^3$

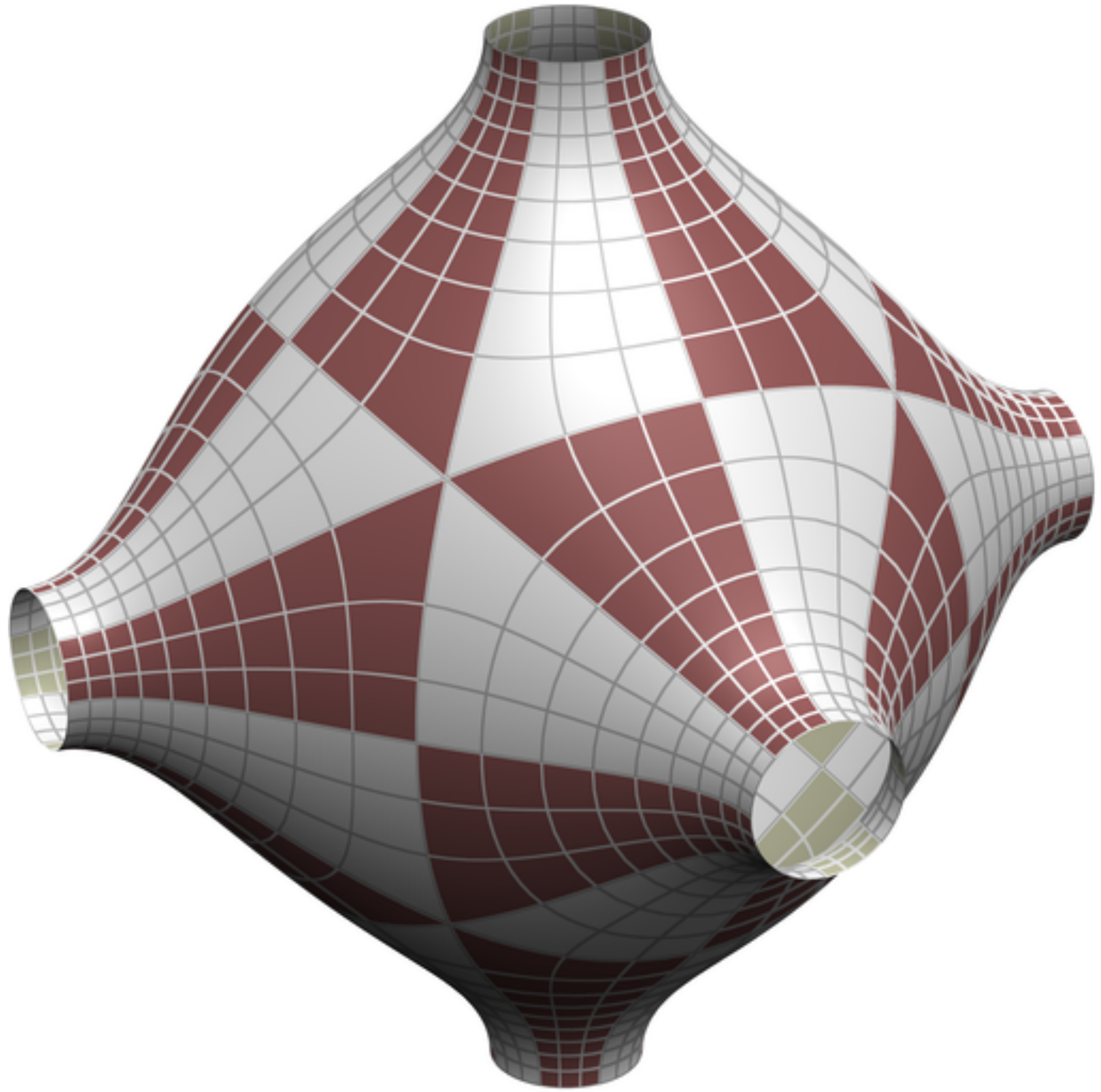
Problem: For each compact Riemann surface find a conformal immersion f that minimizes

$$W(f) = \int_M H^2$$

Critical points: Constrained Willmore surfaces

CMC surfaces in $\mathbb{R}^3$





Images by Nicholas Schmitt

CMC-1 surfaces in H^3

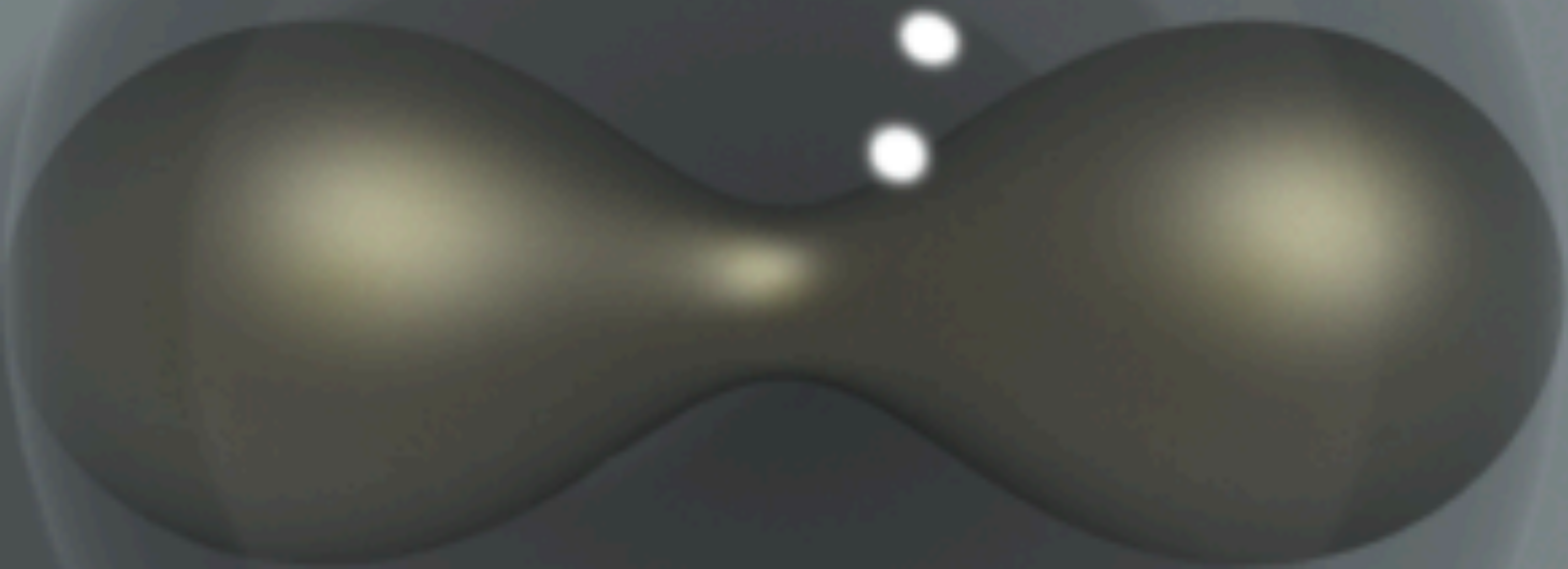
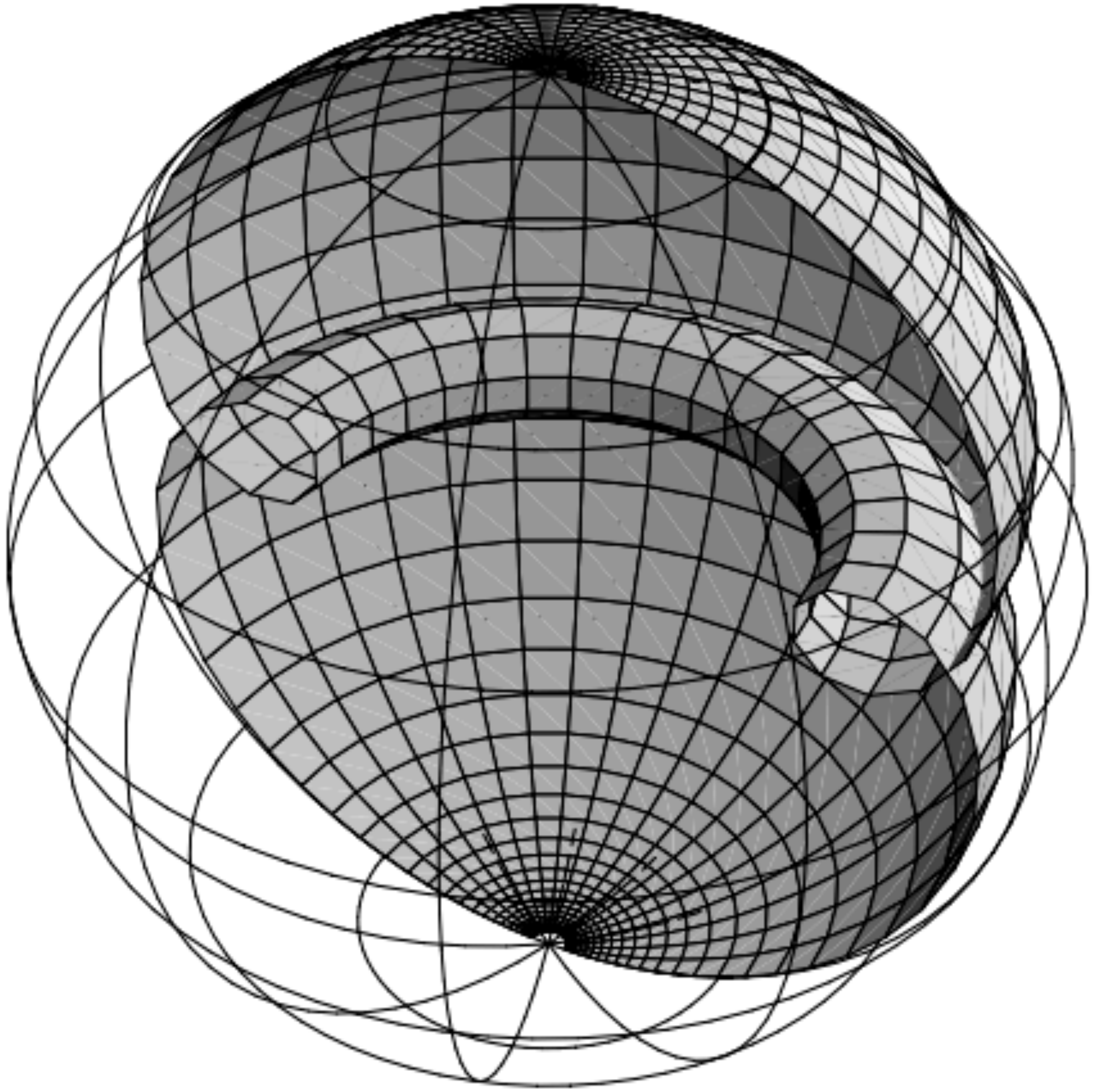


Image by Nicholas Schmitt

CMC-1 surfaces in H^3



CMC-1 surfaces in H^3

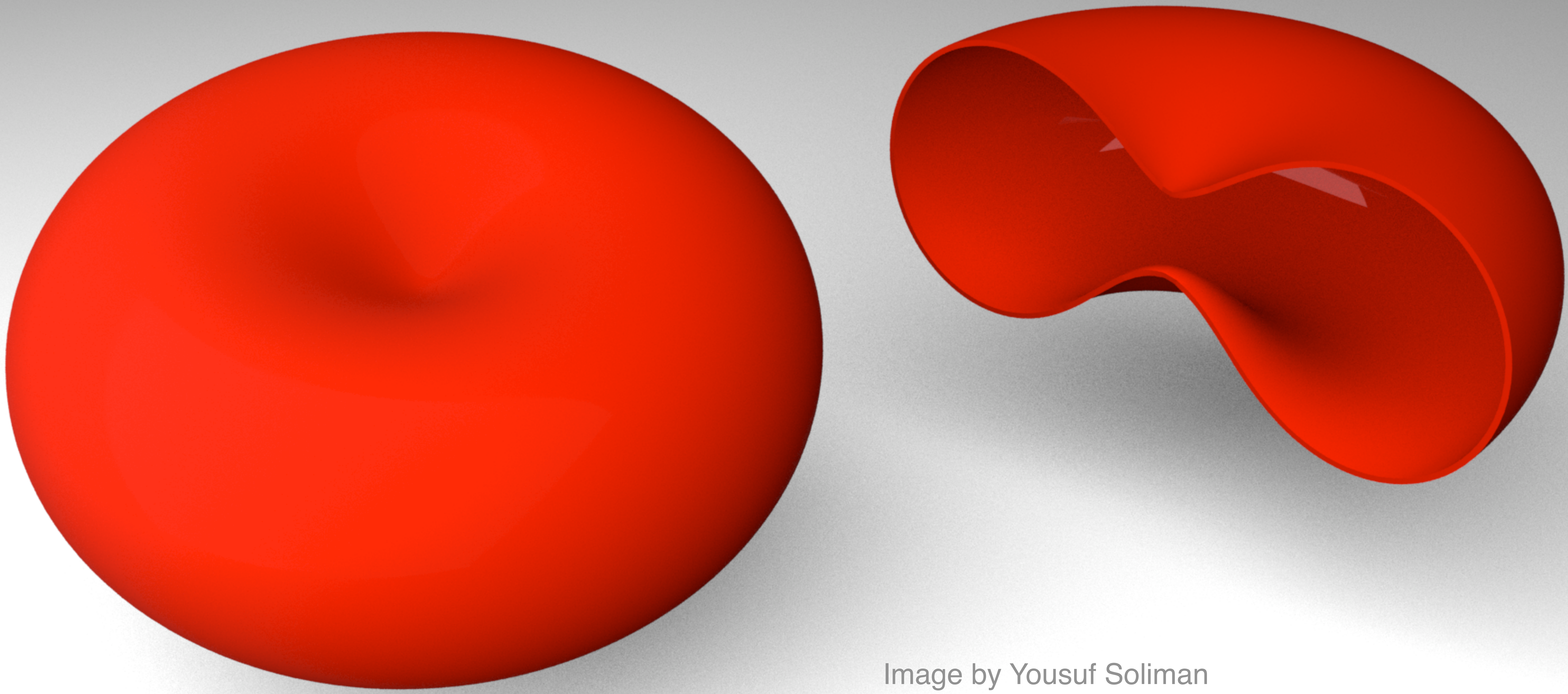


Image by Yousuf Soliman

CMC surfaces in S^3

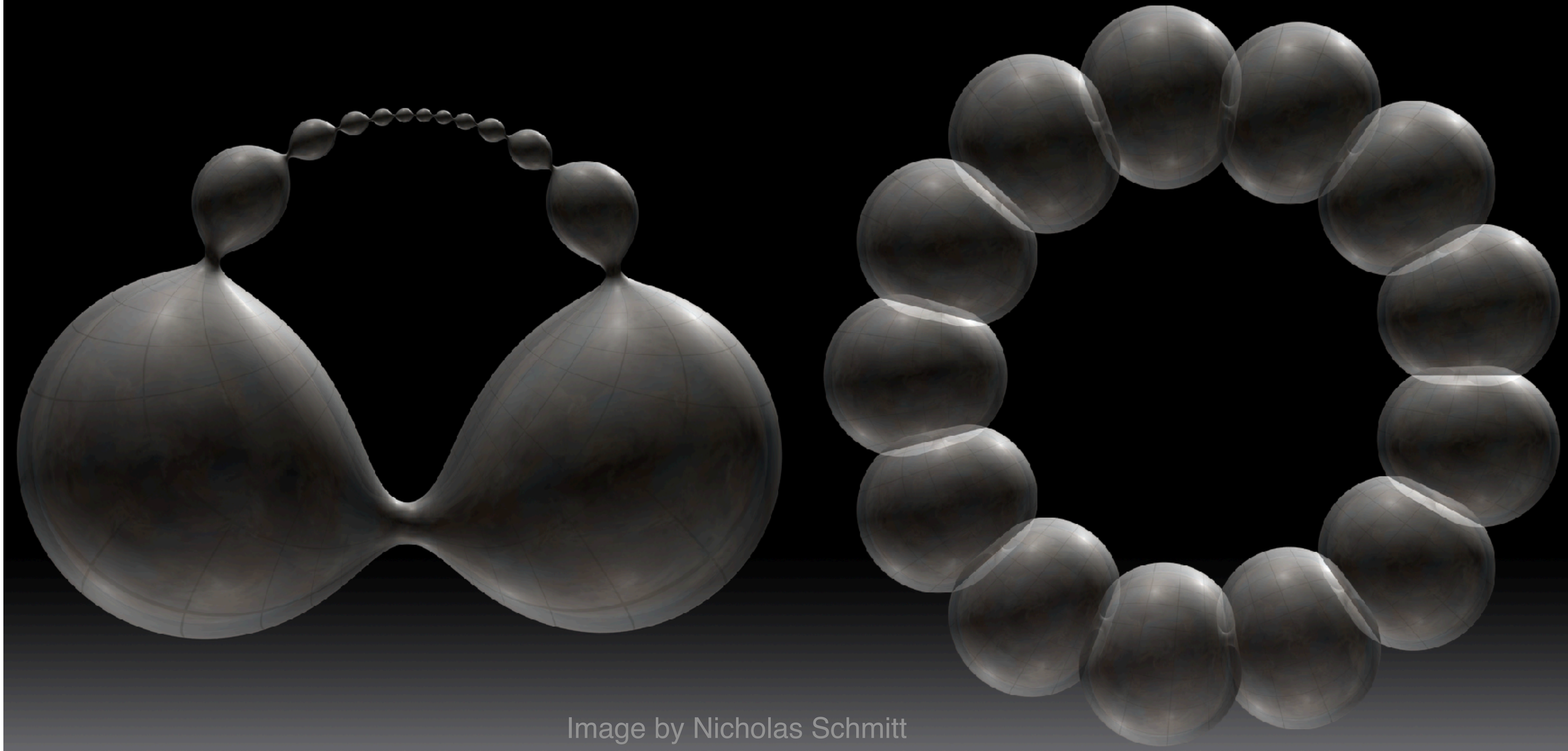


Image by Nicholas Schmitt

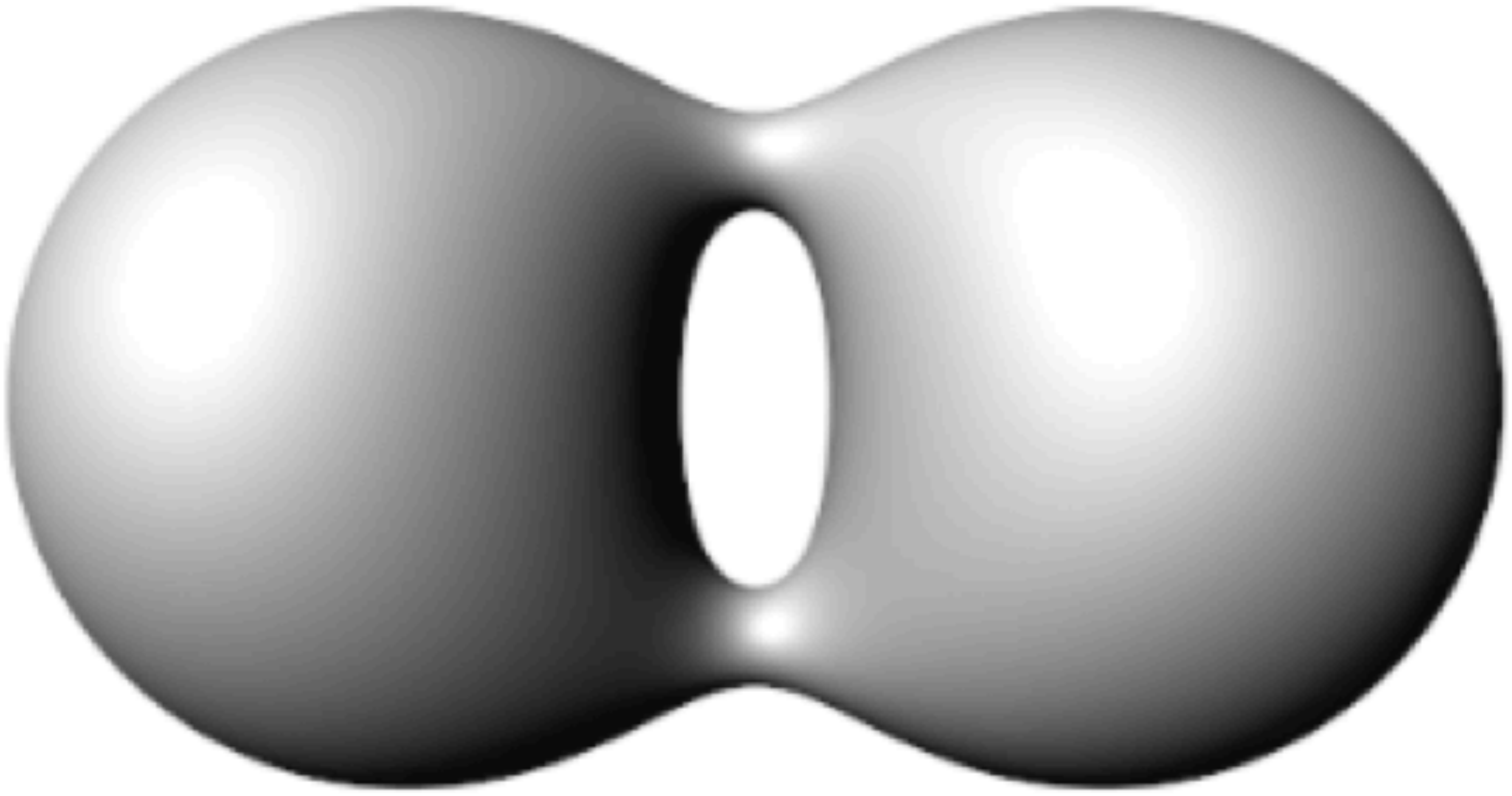
Might be a minimizer of

$$W(f) = \int_M H^2$$

among all conformal immersions

$$f\colon M \to \mathbb{R}^3$$

CMC surfaces in S^3



From: Heller and Pedit 2017

M an oriented surface

$\mathcal{M} = \{J \in \Gamma(\mathrm{End}(TM)\ J^2 = -1\}$ space of conformal structures on M

$T_J\mathcal{M} = \{\dot{J} \in \Gamma\mathrm{End}(TM) \mid \dot{J}J = -J\dot{J}\}$

$\mathcal{M}/_{\mathrm{Diff}_0(M)}$ Teichmüller space of M

$T_{[J]}\left(\mathcal{M}/_{\mathrm{Diff}_0(M)}\right) = T_J\mathcal{M}/_{\{\mathcal{L}_X J \mid X \in \Gamma(TM)\}}$

$$T_{[J]}\left(\mathcal{M}/_{\mathrm{Diff}_0(M)}\right) = T_J\mathcal{M}/_{\{\mathcal{L}_X J \;\mid\; X\in\Gamma(TM)\}}$$

$$T_{[J]}\left(\mathcal{M}/_{\mathrm{Diff}_0(M)}\right) = T_J\mathcal{M}/_{\{\mathcal{L}_X J \mid X\in\Gamma(TM)\}}$$

Constraints on $[J]$ give rise to Lagrange multipliers in the cotangent bundle of Teichmüller space:

$$T^*_{[J]}\left(\mathcal{M}/_{\mathrm{Diff}_0(M)}\right) = \left\{ q \in T^*_J\mathcal{M} \;\middle|\; \begin{array}{l} \langle q \mid \mathcal{L}_X J\rangle = 0 \\ \\ \text{for all } X \in \Gamma(TM) \end{array} \right\}$$

Given J, a field $q \in \Gamma\mathrm{sym}(TM)$ of symmetric bilinear forms is called a **quadratic differential** if for all $X \in T_pM$

$$q(JX, JX) = -q(X, X)$$

For a quadratic differential q and $\dot{J} \in T_J\mathcal{M}$ we can define

$$\langle q \mid \dot{J} \rangle = \int_M (X, Y \mapsto q(\dot{J}X, Y))$$

In this way,

$$T_J^*\mathcal{M} = \{\text{quadratic differentials } q\}$$

and we can define

$$T_{[J]}^*\left(\mathcal{M}/_{\mathrm{Diff}_0(M)}\right) = \left\{ q \in T_J^*\mathcal{M} \;\middle|\; \begin{array}{c} \langle q \,|\, \mathcal{L}_X J \rangle = 0 \\ \\ \text{for all } X \in \Gamma(TM) \end{array} \right\}$$

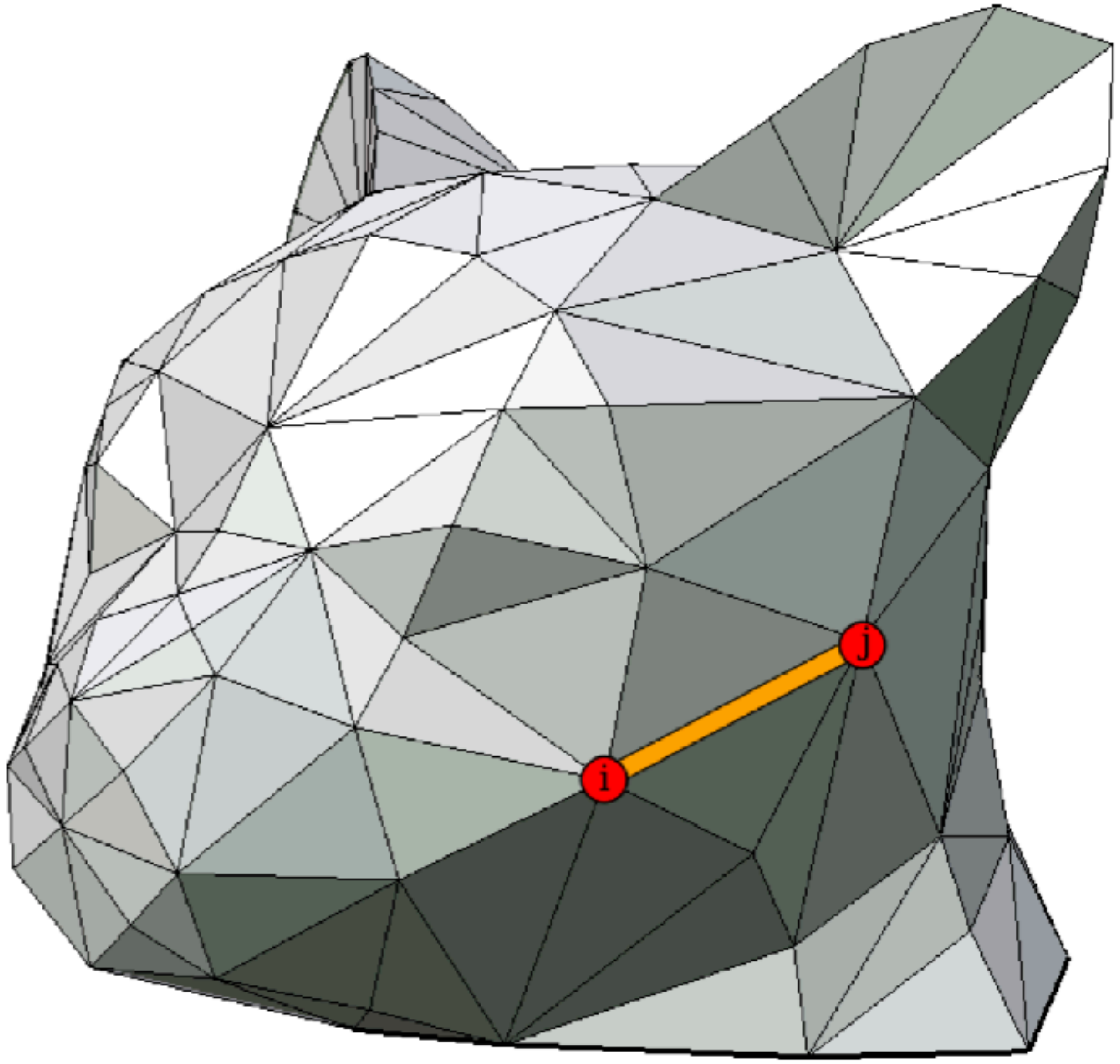
$$:= \{\text{holomorphic quadratic differentials}\}$$

A discrete metric

prescribes a length

$$\ell_{ij} > 0$$

for each edge $ij \in E$

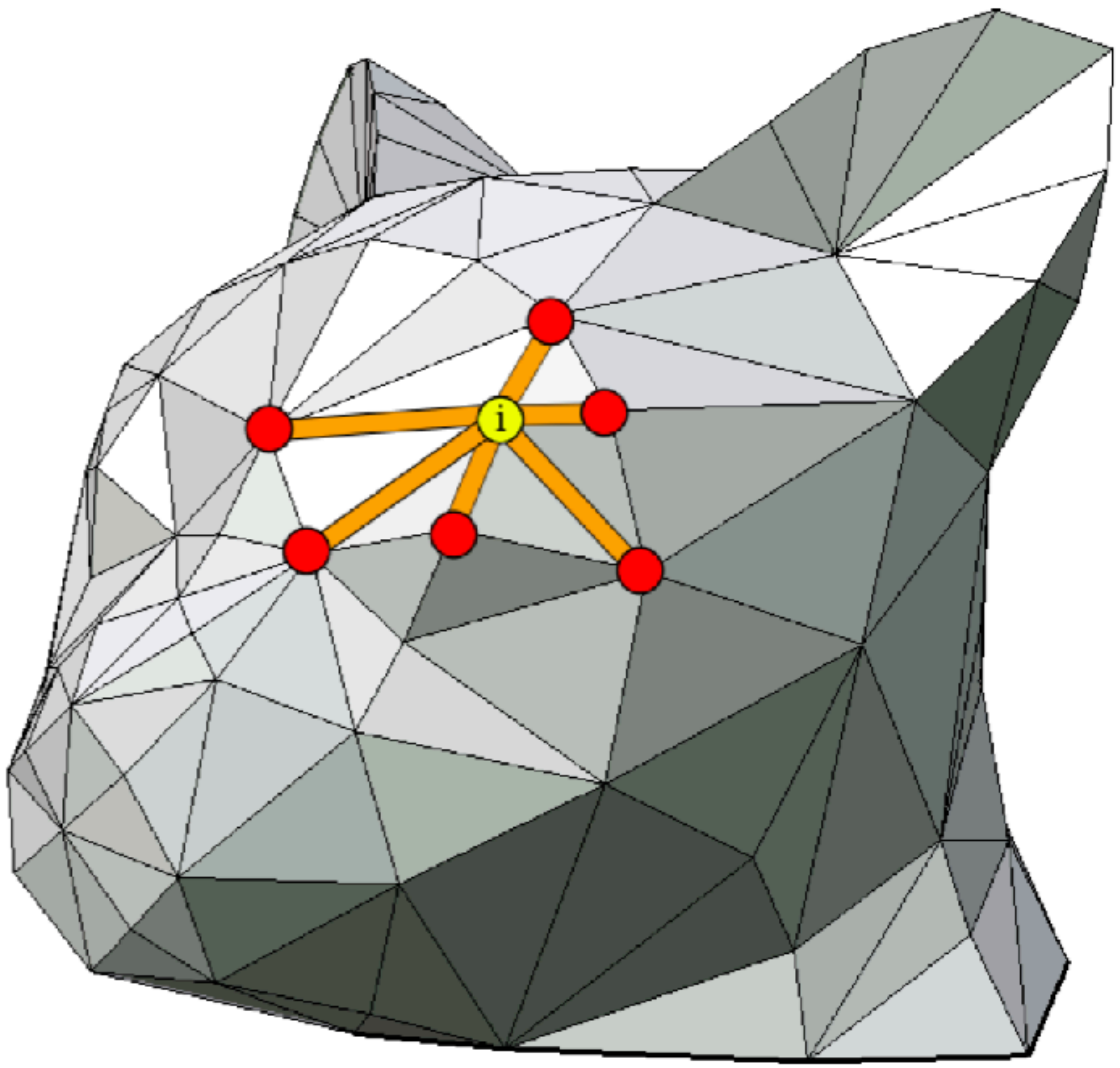


Conformal factors

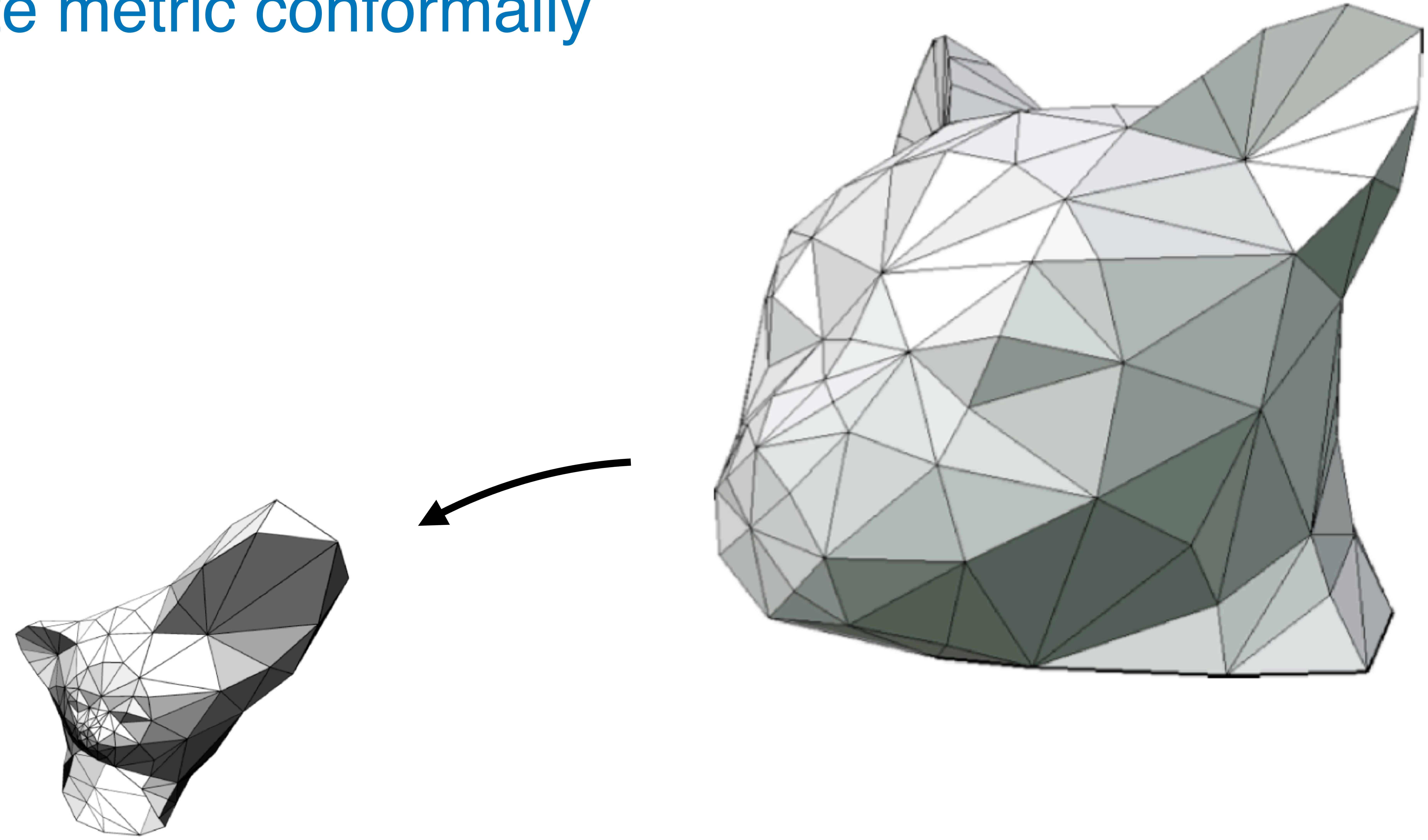
$$e^{u_i}$$

at the vertices change the metric conformally to

$$\tilde{\ell}_{ij} = e^{\frac{u_i + u_j}{2}} \ell_{ij}$$



Möbius transformations change the discrete metric conformally



Suppose we work with some discrete version W of the Willmore functional whose gradient at each vertex $i \in V$ is given by a vector

$$(\operatorname{grad} W)_i \in \mathbb{R}^3$$

In the smooth case, $\operatorname{grad} W$ would be the normal vector field

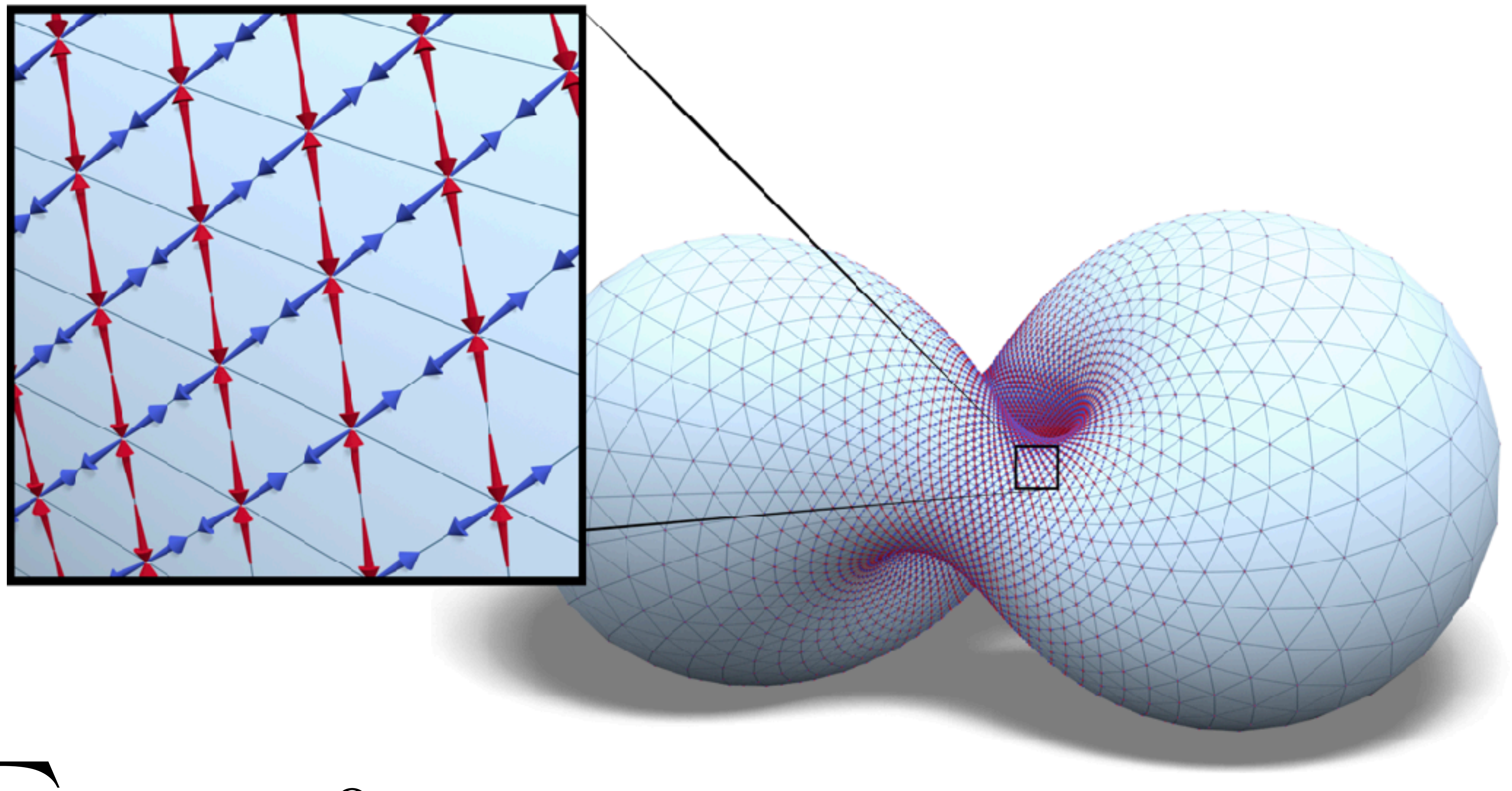
$$\operatorname{grad} W = \big((\Delta H + 2H(H^2 - K)\big)\, N$$

Theorem: A discrete surface $i \mapsto f_i$ is constrained Willmore if and only if there are numbers q_{ij} indexed by the edges $ij \in E$ such that

$$\sum_{ij \in E} q_{ij} = 0$$

$$(\operatorname{grad} W)_i = \sum_{ij \in E} q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}$$

Conformal stress tensor



No resistance to scaling

$$\sum_{ij \in E} q_{ij} = 0$$

$$(\operatorname{grad} W)_i = \sum_{ij \in E} q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}$$

Force equilibrium

$$\sum_{ij \in E} q_{ij} = 0$$

$$(\operatorname{grad} W)_i = \sum_{ij \in E} q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}$$

If $f_i \in \mathbb{R}^2 \subset \mathbb{R}^3$ for all $i \in V$ then $(\operatorname{grad} W)_i = 0$

$$\sum_{ij \in E} q_{ij} = 0$$

$$0 = \sum_{ij \in E} q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}$$

If $f_i \in \mathbb{R}^2 \subset \mathbb{R}^3$ for all $i \in V$ then $(\operatorname{grad} W)_i = 0$ and

$$\sum_{ij \in E} q_{ij} = 0$$

$$0 = \sum_{ij \in E} q_{ij} \frac{f_j - f_i}{|f_j - f_i|^2}$$

says that $ij \mapsto q_{ij}$ is a discrete holomorphic quadratic differential (Lam 2016)

Our algorithm uses a recent new approach to constrained optimization called Competitive Gradient Descent

Inspired by game theory:

Player 1 cares about the conformality constraint

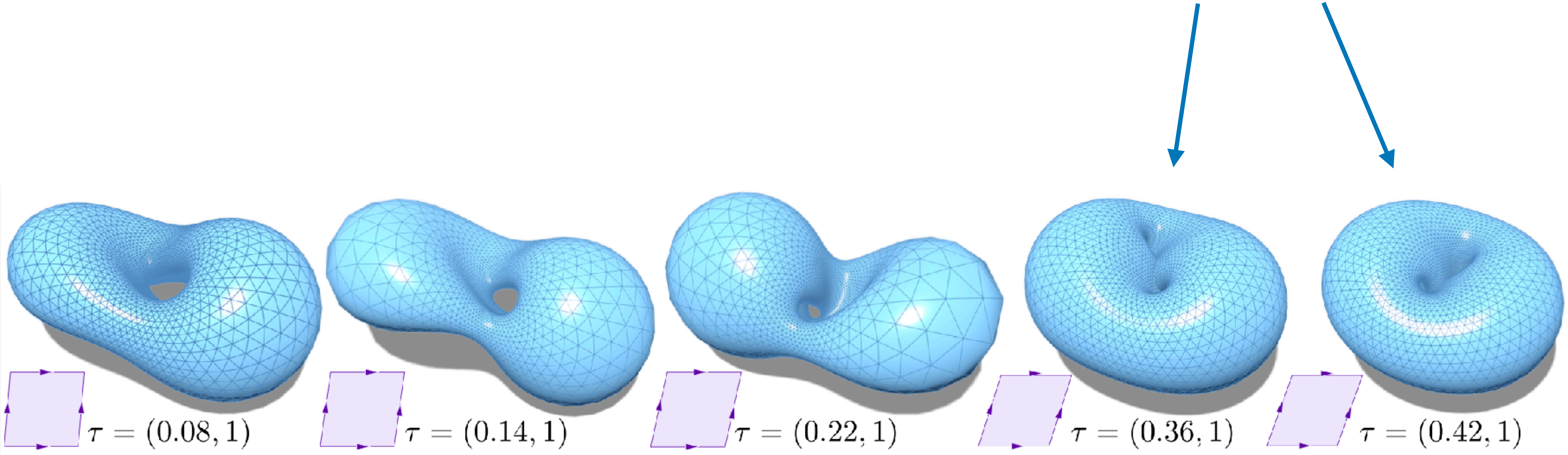
Player 2 cares about Willmore minimisation

Results: Tori near the Clifford torus

Equivariant constrained Willmore tori found by Heller and Ndiaye (Images by Nicholas Schmitt)

Minimizers found numerically by our algorithm (Images by Yousuf Soliman)

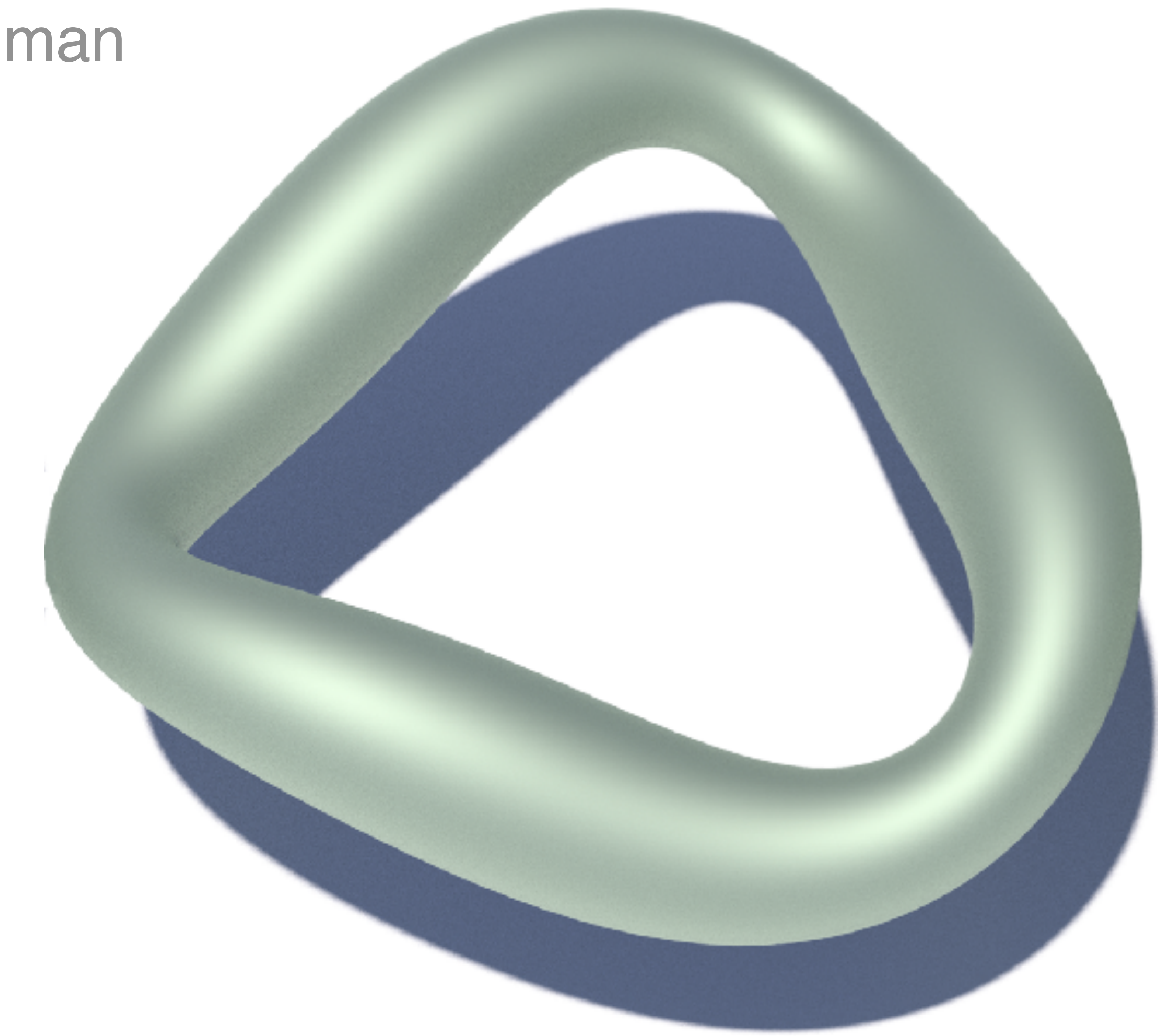
For larger twist we get self-intersections



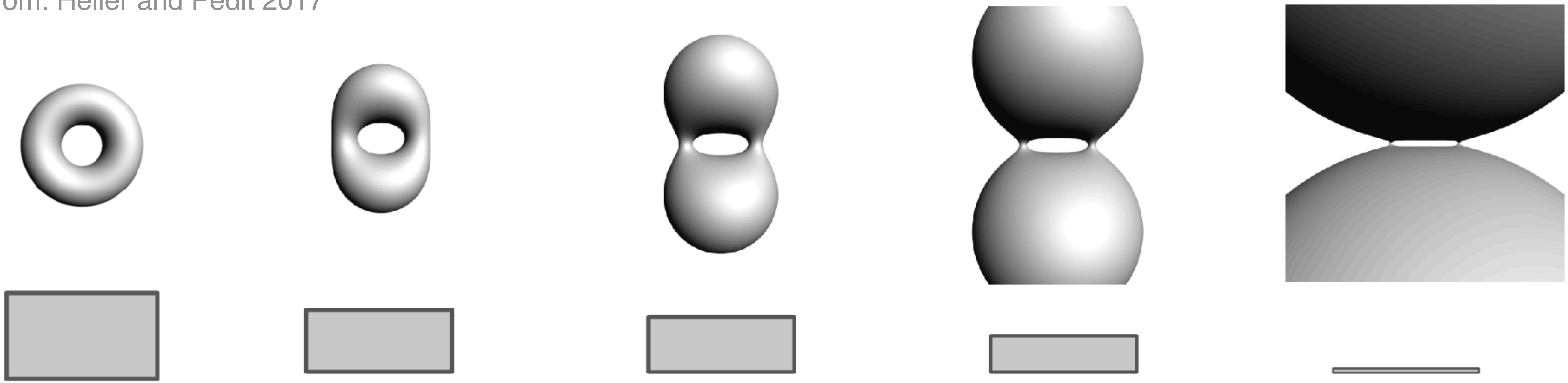
Incompatible with equivariance

Twisting can stabilize thin tori

Image by Yousuf Soliman



From: Heller and Pedit 2017



Results: higher genus

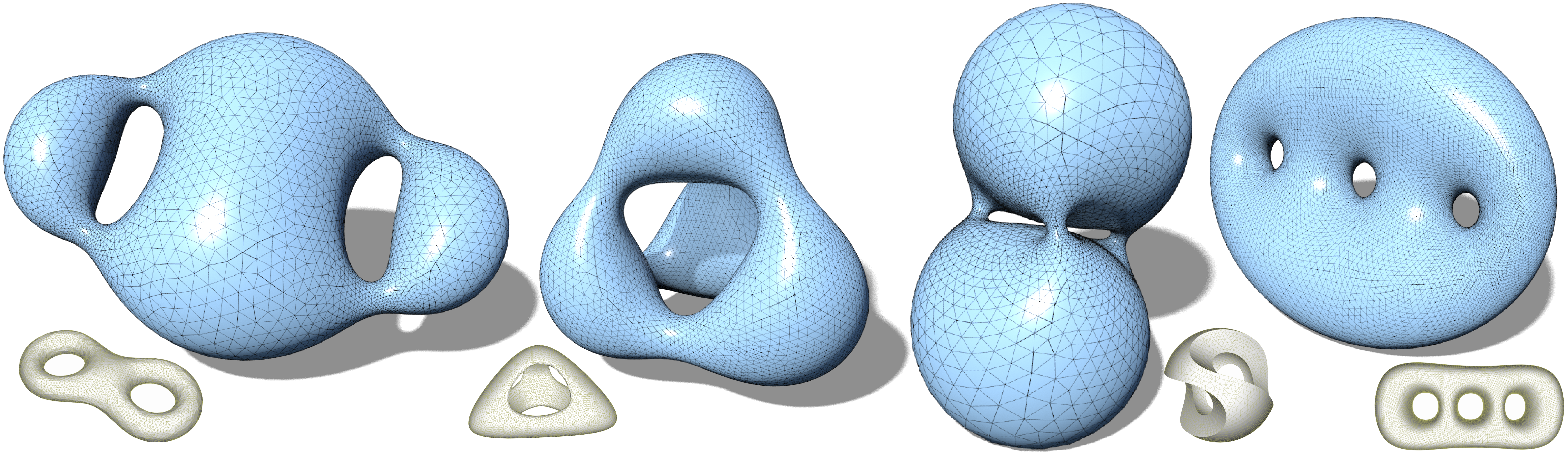


Image by Yousuf Soliman

Image by Yousuf Soliman

Tori with a translational period

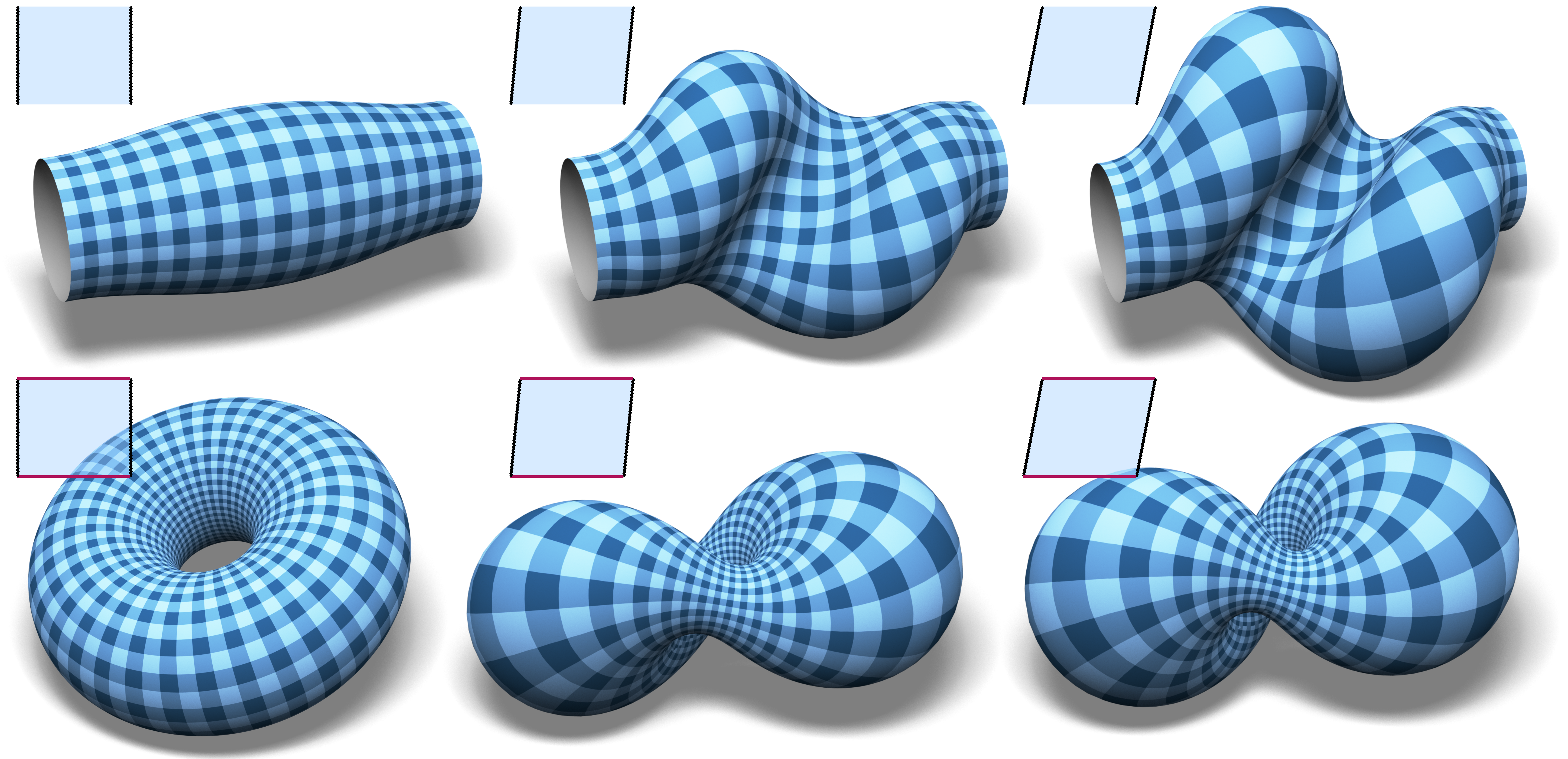


Image by Yousuf Soliman

Point constraints

Prescribing the position $f(p_i)$ for finitely many points $p_1, \ldots, p_n \in M$ is a well-posed problem.

Without conformality constraint, minimising $W(f)$ while prescribing $f(p_1), f(p_2), f(p_3), f(p_4)$ for

immersions $f\colon S^2 \to \mathbb{R}^3$
always leads to a round sphere.

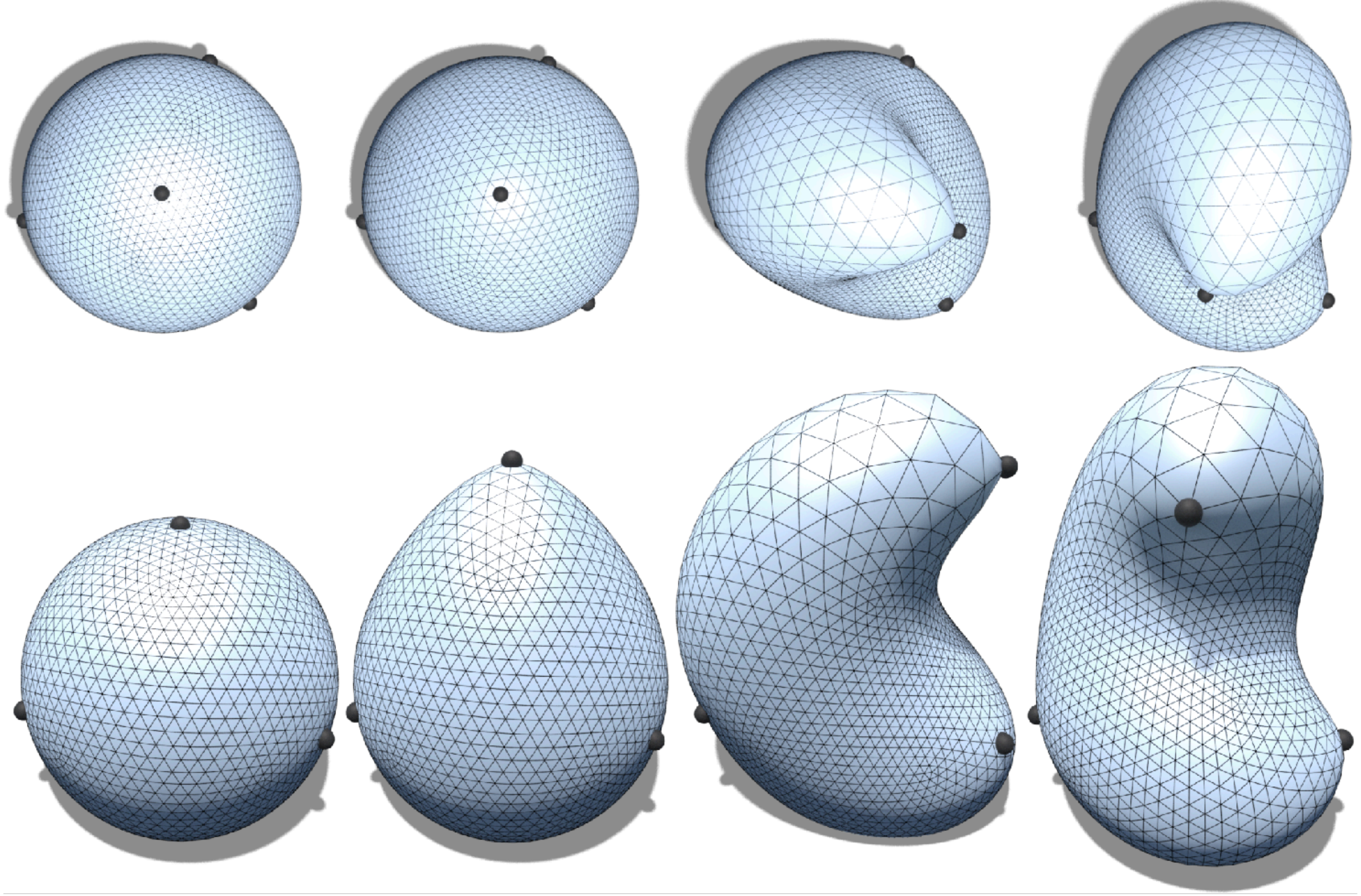


Image by Yousuf Soliman

One even can even prescribe the conformal factor e^{2u} at $p_1, \ldots, p_n$.

CMC-1 spheres in H^3 solve such a boundary value problem.

Images by Nicholas Schmitt

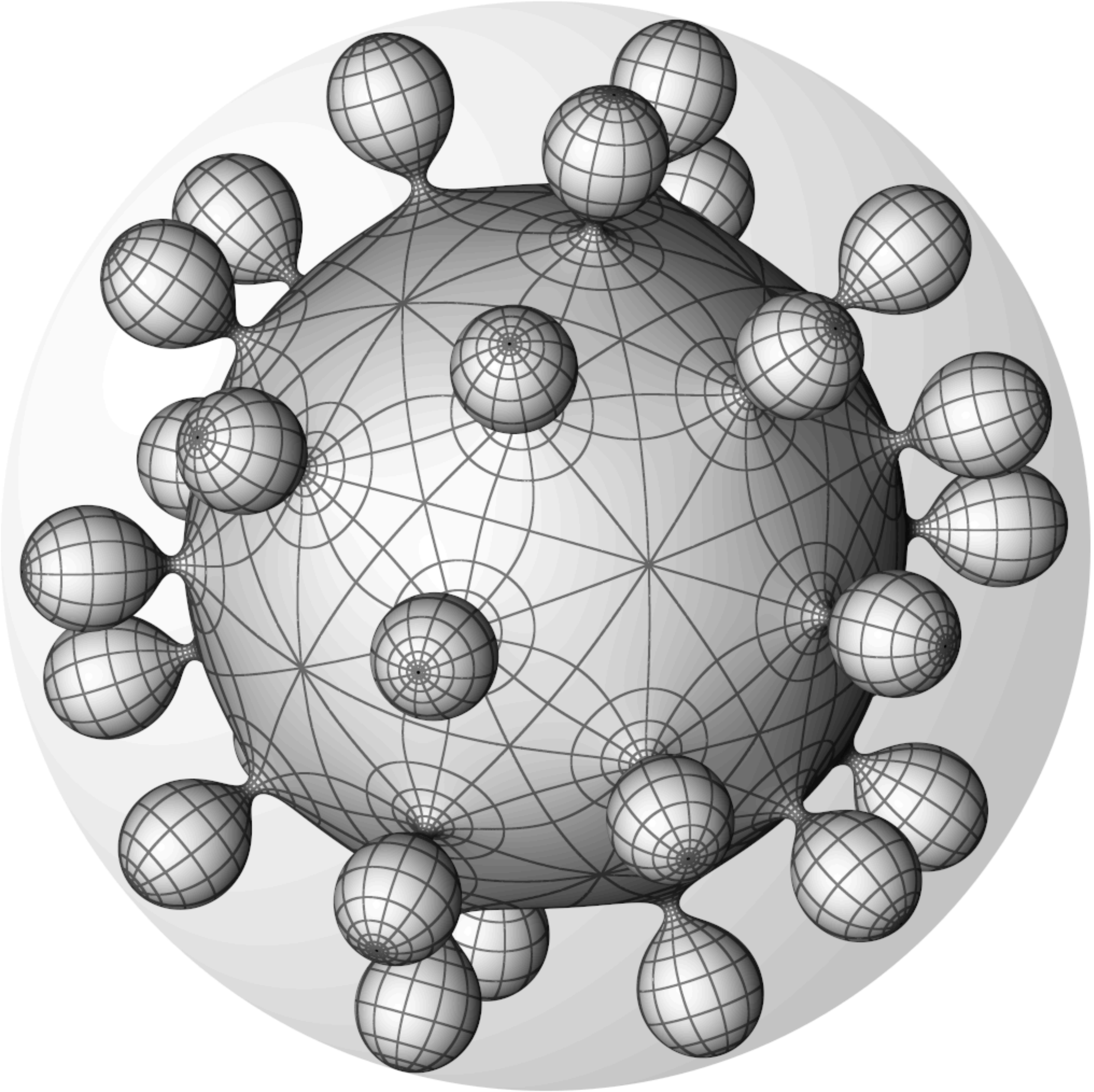
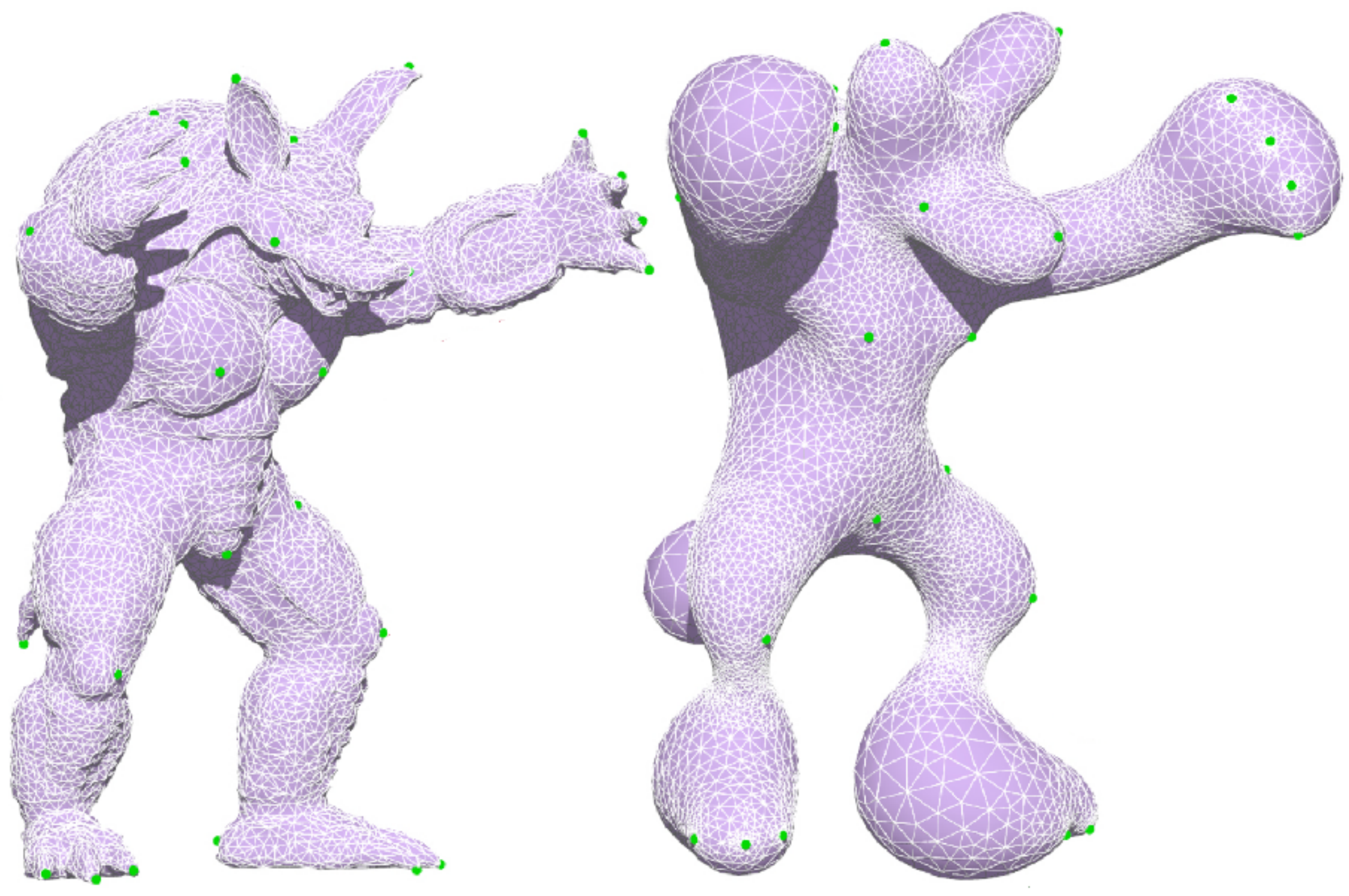
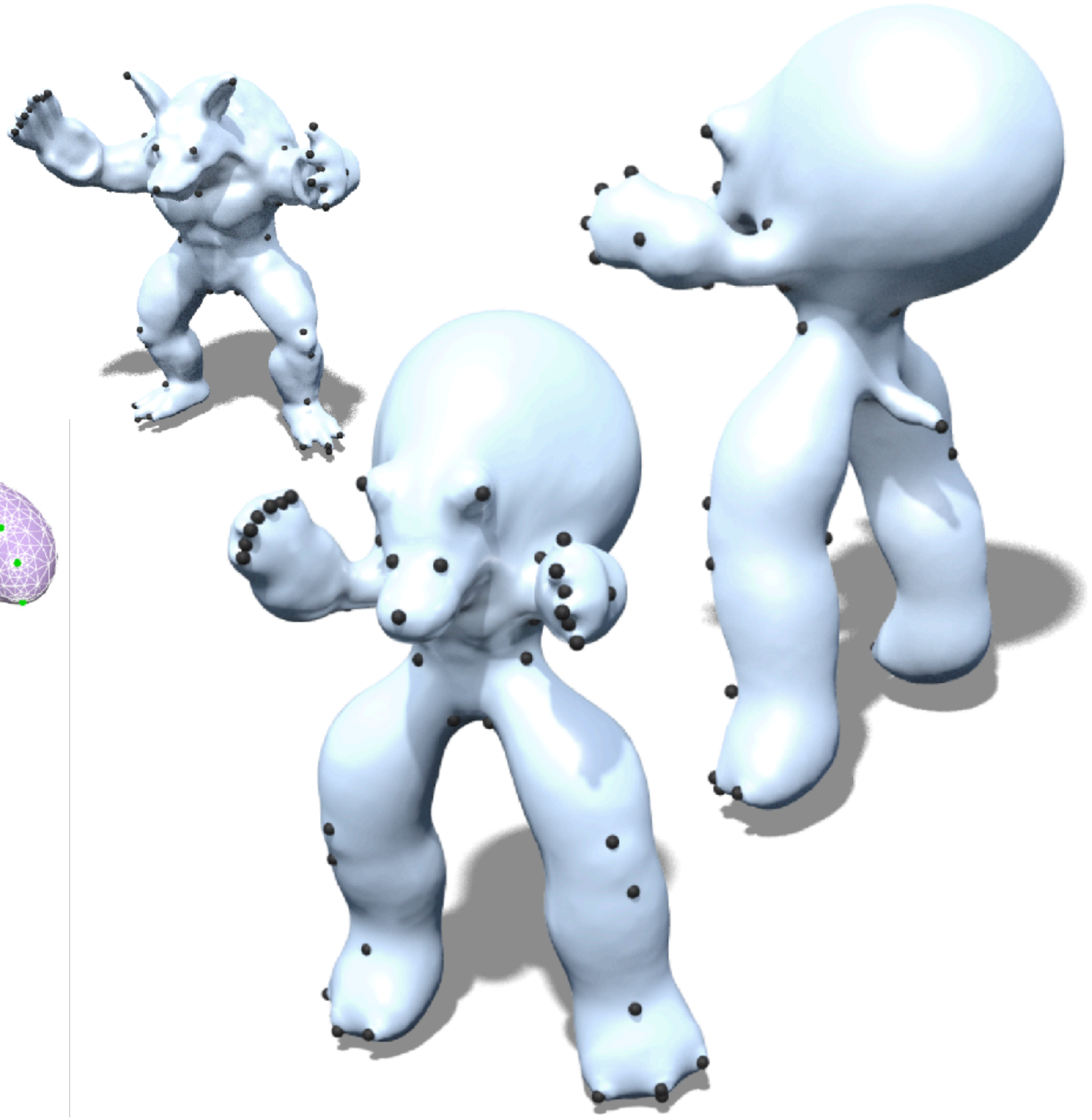


Image by Nicholas Schmitt

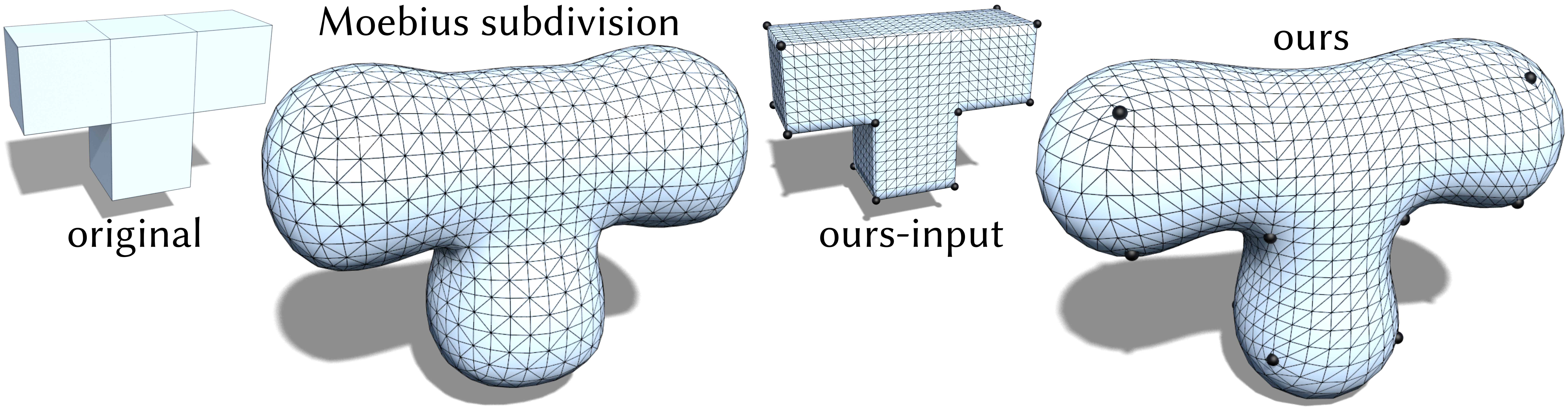
Vaxman, Müller and Weber

Regular Meshes from Polygonal Patterns





Constraint Willmore with point constraints



Vaxman, Müller and Weber

Canonical Möbius Subdivision

Constraint Willmore with point constraints

Thank You!

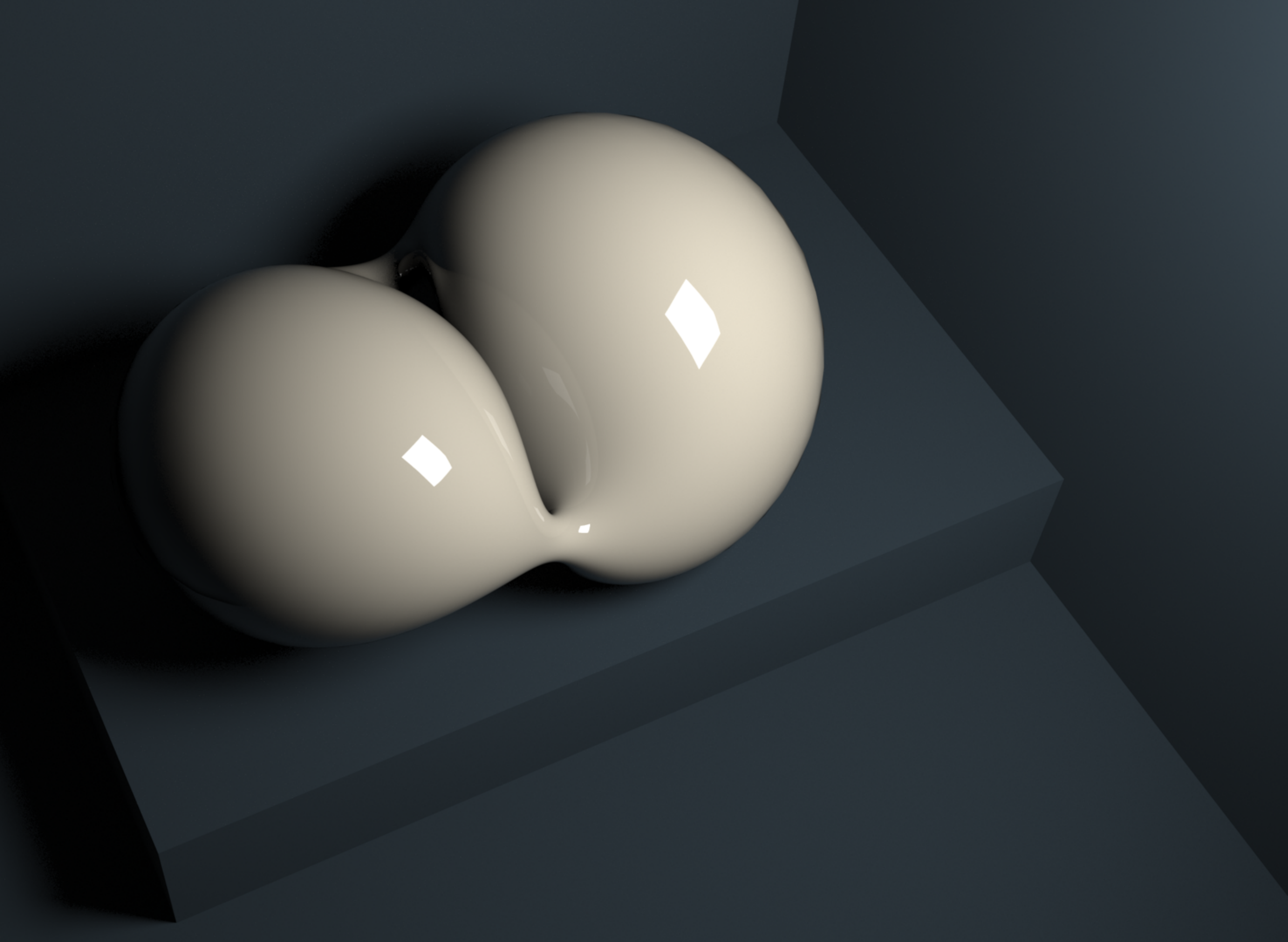


Image by Yousuf Soliman