Triangulated ternary disc packings that maximize the density

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supervised by

Thomas Fernique

Conference on Digital Geometry and Discrete Variational Calculus

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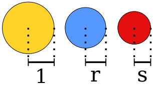
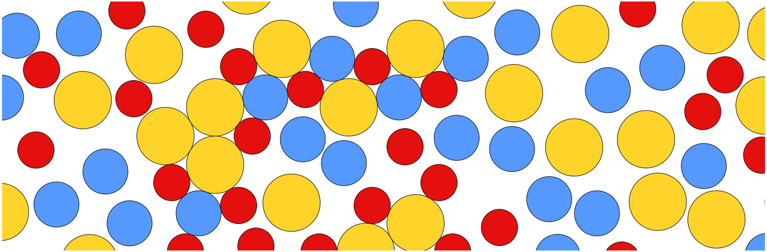
Discs:



Packing P:
(in R^2)



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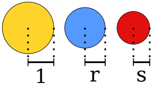

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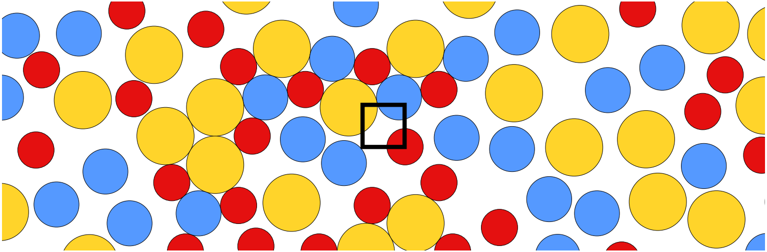

Density:

$$\delta(P) = \limsup_{n\to\infty} \frac{area([-n,n]^2 \cap P)}{area([-n,n]^2)}$$

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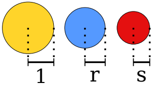

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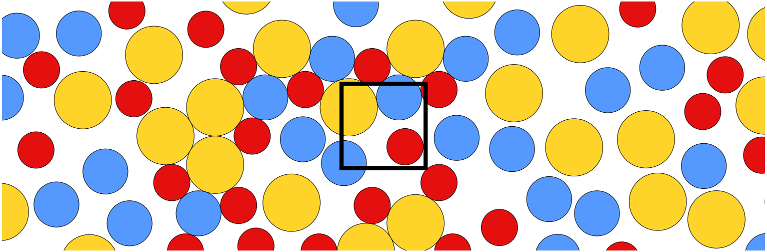

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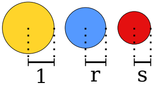

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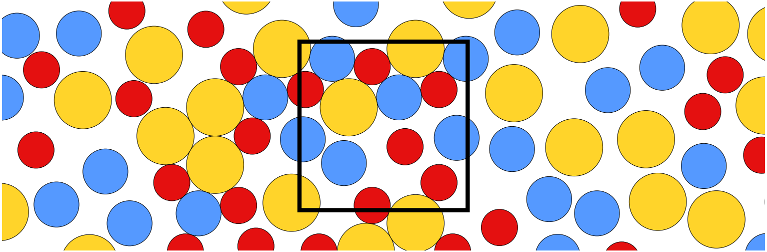

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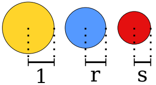

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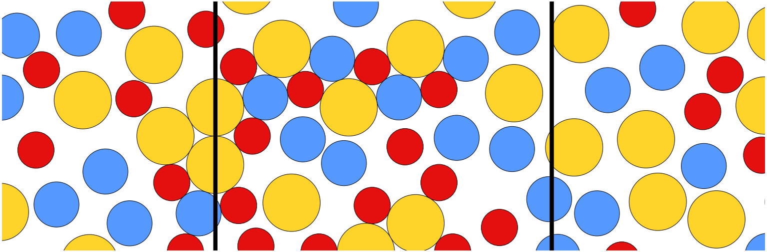

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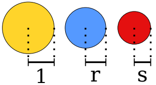

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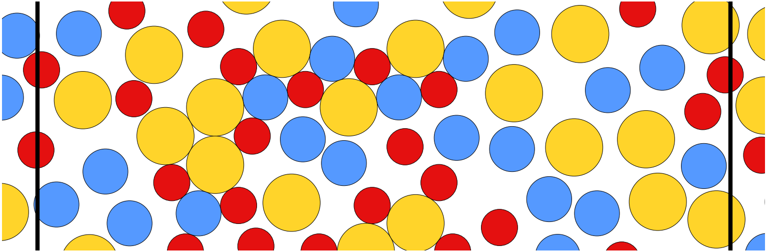

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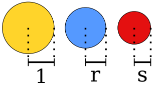

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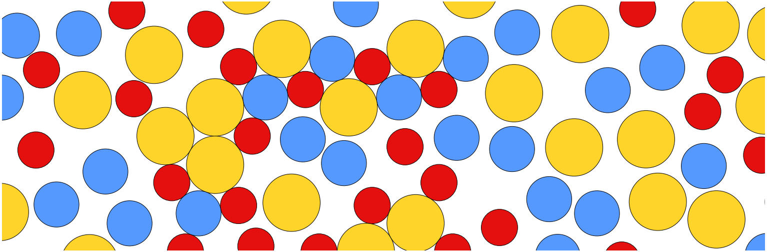

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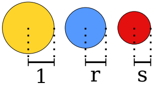

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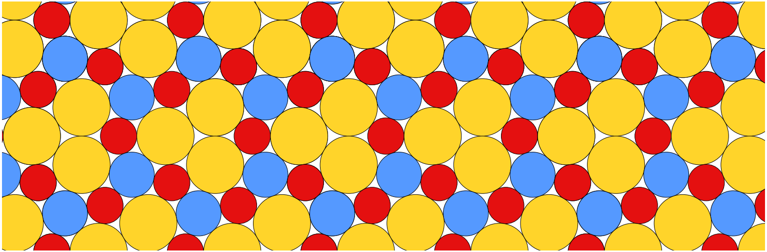

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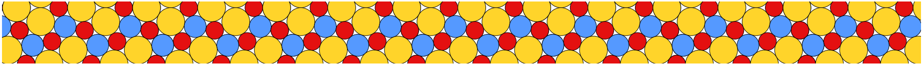
Which packings maximize the density?

A packing is called **triangulated** if each "hole" is bounded by three tangent discs:



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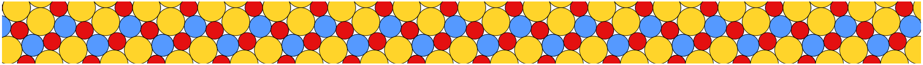
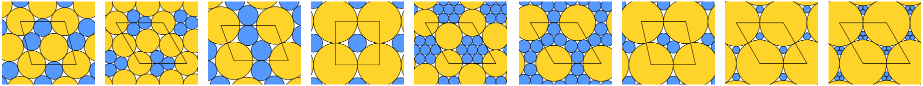


Kennedy, 2006

There are 9 values of r allowing triangulated packings.



Triangulated packings

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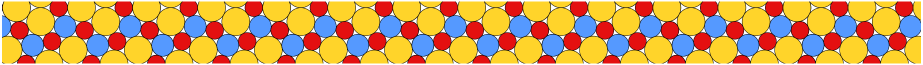

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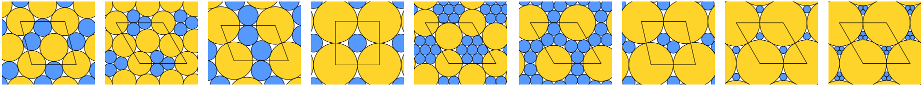
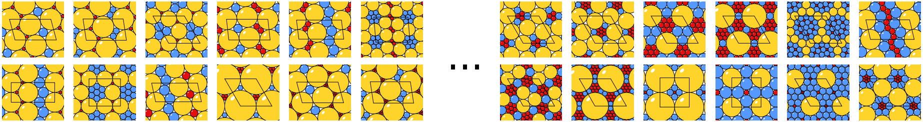


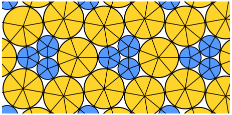
Fernique, Hashemi, Sizova 2019

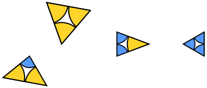
There are 164 pairs (r, s) allowing triangulated packings.

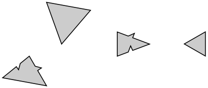


Tilings

triangulated packings $\sim$ tilings by triangles with local rules







density = weighted proportion of tiles

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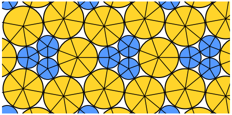
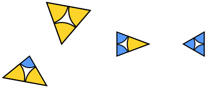
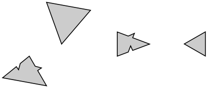






density = weighted proportion of tiles

Conformal maps

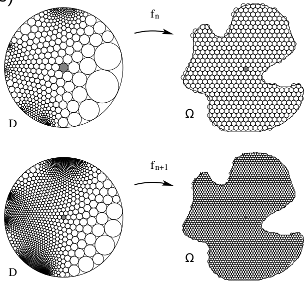
(preserve angles between curves)

- $\exists$ between any pair of open topological discs
- may be hard to construct

Conjecture (Thurston, 1985)

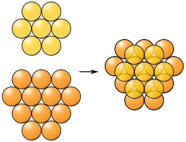
Circle packings can be used to approximate conformal mappings.

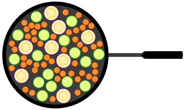
Proof: Rodin, Sullivan, 1987



K. Stephenson, Approximation of conformal structures via circle packing, Computational Methods and Function Theory, 1997

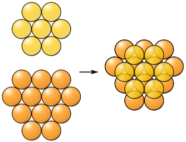
- Packing fruits and vegetables

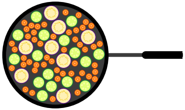




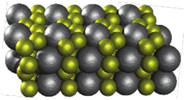
Motivation: real life

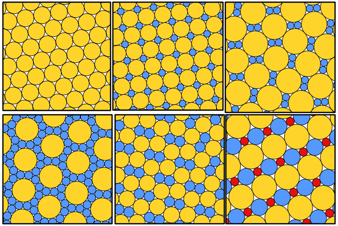
- Packing fruits and vegetables

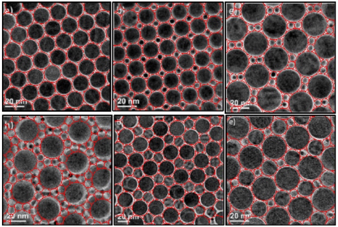




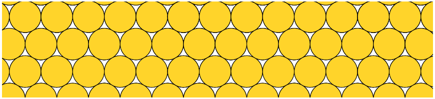
- Making compact materials







Binary and ternary superlattices self-assembled from colloidal nanodisks and nanorods. Journal of the American Chemical Society, 137(20):6662–6669, 2015.

2D hexagonal -packing:  $\delta = \frac{\pi}{2\sqrt{3}}$

Lagrange, 1772

Hexagonal packing maximize the density among lattice packings.

Thue, 1910 (Toth, 1940)

Hexagonal packing maximize the density.

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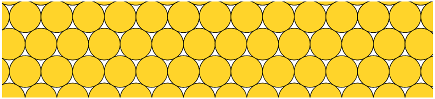

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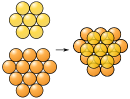


$\delta = \frac{\pi}{3\sqrt{2}}$

Gauss, 1831

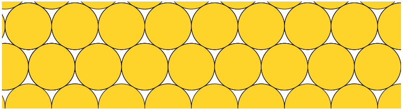
Hexagonal packing maximize the density among lattice packings.

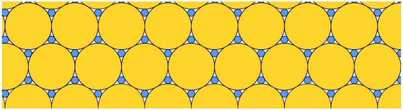
Hales, Ferguson, 1998–2014 (Conjectured by Kepler, 1611)

Hexagonal packing maximize the density.

Two discs of radii 1 and r:

Lower bound on the density: $\frac{\pi}{2\sqrt{3}}$ (hexagonal packing with only 1 disc used)





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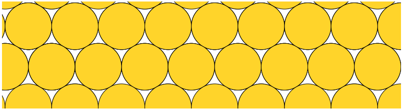
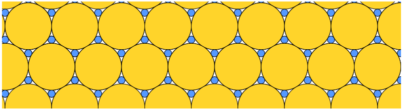
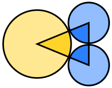




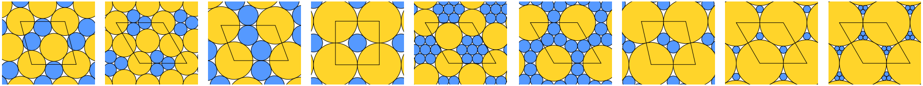
Upper bound on the density:

Florian, 1960

The density of a packing never exceeds the density in the following triangle:



Heppes 2000,2003; Kennedy 2004; Bedaride, Fernique, 2019



All these 9 triangulated packings maximize the density.

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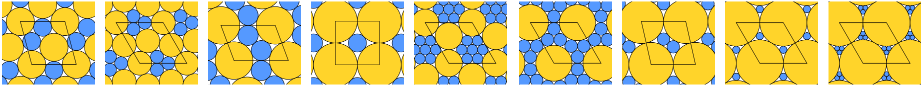

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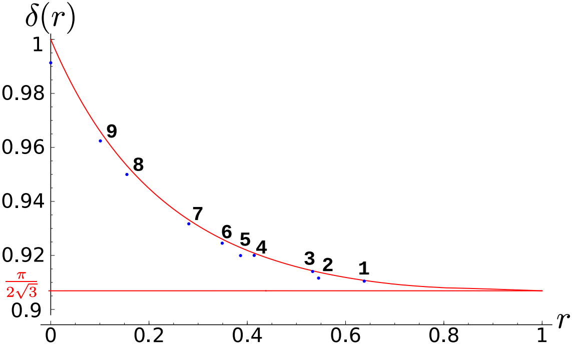
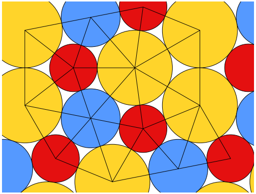
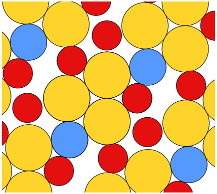
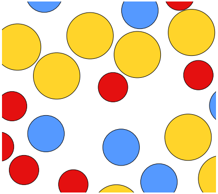


Conjecture (Connelly, 2018)

If a finite set of discs allows a **saturated** triangulated packing then the density is maximized on a saturated triangulated packing.







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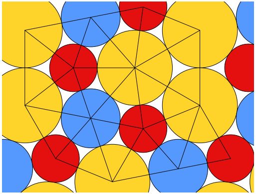
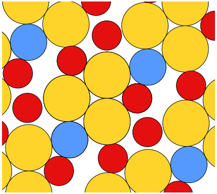
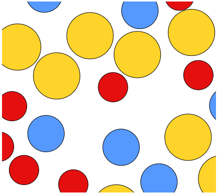
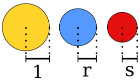




True for and .

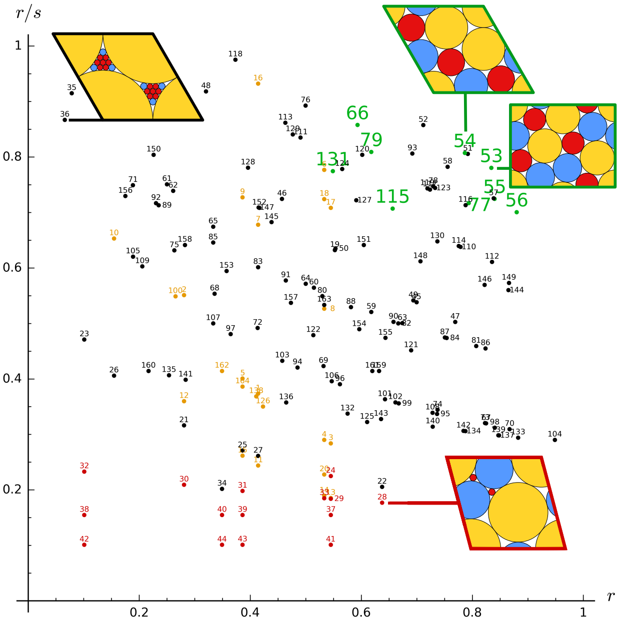
What happens with ?

3 discs:



164 (r, s) allowing triangulated packings:
(Fernique, Hashemi, Sizova 2019)

- 15 cases: non saturated
- 24 cases: a 2-discs packing is denser
- Case 53 is proved
  (Fernique 2019)



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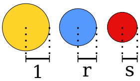

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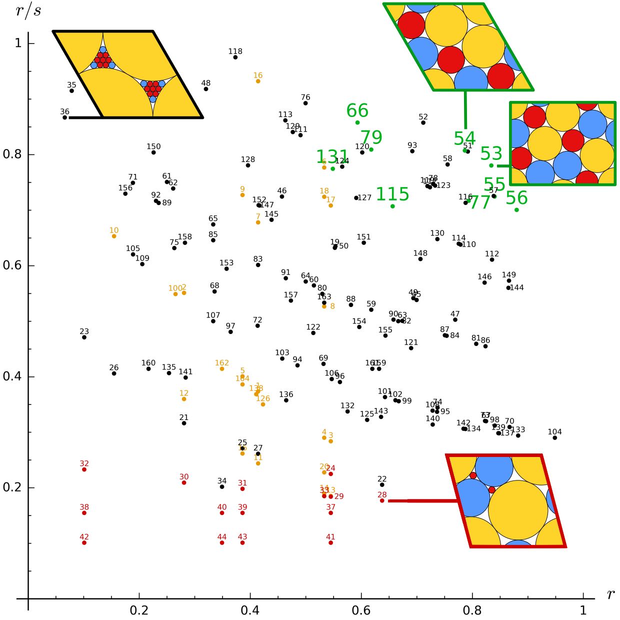

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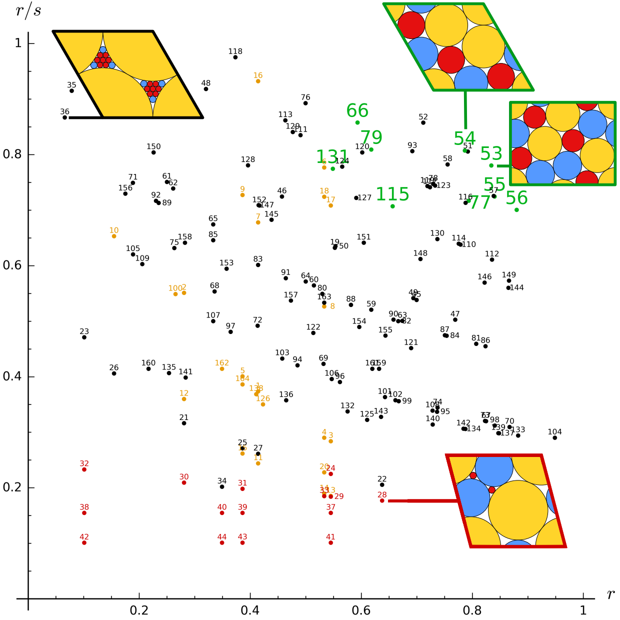
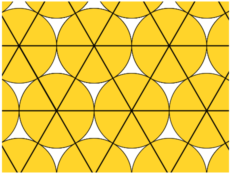
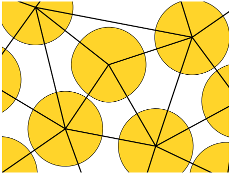


Idea of the proof for

A **Delaunay triangulation** of a packing: no points inside a circumscribed circle



$\delta^* = \delta_{\triangle^*} = \frac{\pi}{2\sqrt{3}}$



$\forall \triangle, \ \delta_\triangle \leq \delta_{\triangle^*} = \delta^*$

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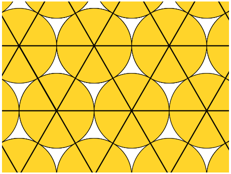

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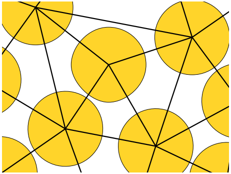

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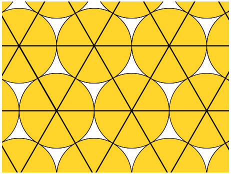
A **Delaunay triangulation** of a packing: no points inside a circumscribed circle



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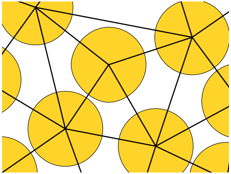

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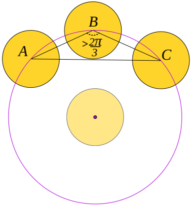

- The largest angle of any $\triangle$ is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

$\hat{A} \leq \frac{\pi}{6} \Rightarrow R = \frac{|BC|}{2 \sin \hat{A}} \geq \frac{1}{\sin \hat{A}} \geq 2$

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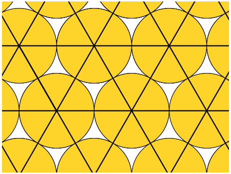

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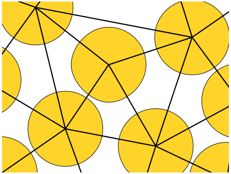

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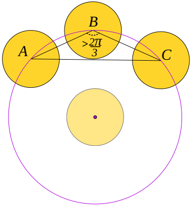

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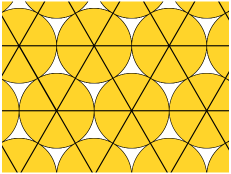

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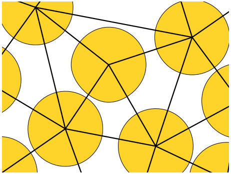

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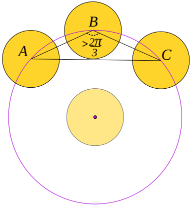

- The largest angle of any $\triangle$ is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ $\qquad \hat{A} \leq \frac{\pi}{6} \Rightarrow R = \frac{|BC|}{2\sin\hat{A}} \geq \frac{1}{\sin\hat{A}} \geq 2$
- The density of a triangle $\triangle$: $\delta_\triangle = \frac{\pi/2}{area(\triangle)}$
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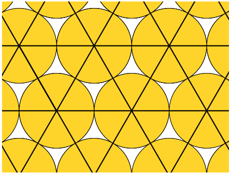
Idea of the proof for

A **Delaunay triangulation** of a packing: no points inside a circumscribed circle



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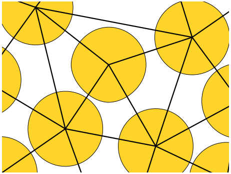

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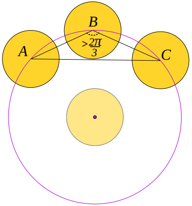

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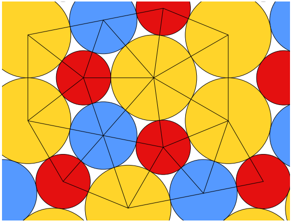
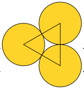
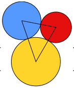
- Thus the density of ABC is less or equal to $\frac{\pi/2}{\sqrt{3}} = \delta_{\triangle^*}$

Delaunay triangulation → weighted by the disc radii

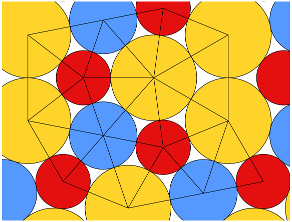


Triangles have different densities:

$\delta($  $) \neq \delta($  $)$

What to do?

Delaunay triangulation $\rightarrow$ weighted by the disc radii



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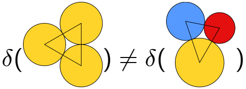

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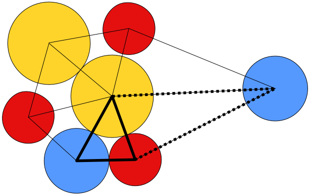
Redistribution of the densities:



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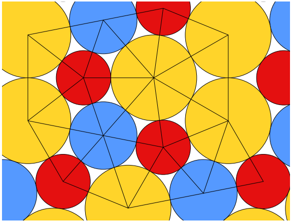

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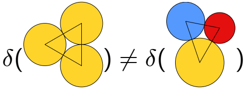

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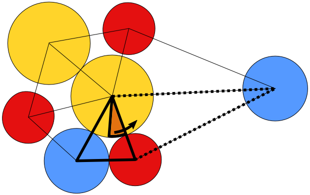
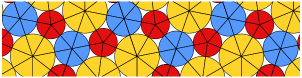
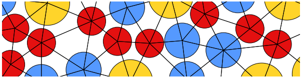


Some triangles “share their density” with neighbors

$\mathcal{T}^*$ – saturated triangulated packing of density δ^*



$\mathcal{T}$ – any other saturated packing with the same discs



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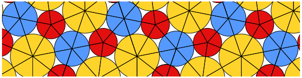

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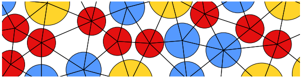


The **sparsity** of a triangle $\triangle \in \mathcal{T}$: $S(\triangle) = \delta^* \times area(\triangle) - cov(\triangle)$

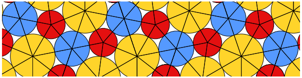
$S(\triangle) > 0$ iff the density of covering of $\triangle$ is less than δ^*

$S(\triangle) < 0$ iff the density of covering of $\triangle$ is greater than δ^*

$$\delta^* \geq \delta(\mathcal{T}) \Leftrightarrow \sum_{\mathcal{T}} S(\triangle) \geq 0$$

Proof for

$\mathcal{T}^*$ – saturated triangulated packing of density δ^*



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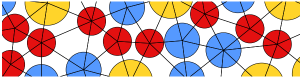

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For this, we introduce a **potential** U such that for any triangle $\triangle \in \mathcal{T}$,

$$S(\triangle) \geq U(\triangle) \qquad (\triangle)$$

and

$$\sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0 \qquad (U)$$

(U): Instead of proving the **global** inequality

$$\sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0 \qquad (U)$$

we decompose $U(\triangle)$ into three vertex potentials: if A, B and C are the vertices of $\triangle$,

$$U(\triangle) = \dot{U}^A_\triangle + \dot{U}^B_\triangle + \dot{U}^C_\triangle$$

and prove a **local** inequality for each vertex $v \in \mathcal{T}$:

$$\sum_{\triangle \in \mathcal{T} | v \in \triangle} \dot{U}^v_\triangle \geq 0 \qquad (\bullet)$$

Proof for ⬤●• : (U)

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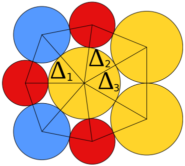
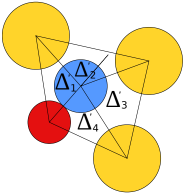


$$4\dot{U}^v_{\triangle_1} + 2\dot{U}^v_{\triangle_2} + \dot{U}^v_{\triangle_3} = 0$$



$$\dot{U}^{v'}_{\triangle'_1} + \dot{U}^{v'}_{\triangle'_2} + \dot{U}^{v'}_{\triangle'_3} + \dot{U}^{v'}_{\triangle'_4} > 0$$

Delaunay triangulation properties → finite number of cases → verification by computer

Inequalities and interval arithmetic

To store and perform computations on transcendental numbers (like π), we use intervals.

A representation of a number x is an interval I whose endpoints are exact values representable in a computer memory and such that $x \in I$.

```
sage: x = RIF(0,1)                                   # Interval [0,1]
sage: x<2
True                                                 # ∀t ∈[0,1], t < 2
sage: (x+x).endpoints()
(0.0, 2.0)                                           # [0,1]+[0,1]
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```
sage: Ipi = RIF(pi)                         # Interval for π
(3.14159265358979, 3.14159265358980)
sage: sin(Ipi).endpoints()                  # Interval for sin(π)
(-3.21624529935328e-16, 1.22464679914736e-16)
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```

Intersecting intervals are incomparable:

```
sage: sin(Ipi)<=0
False
sage: sin(Ipi)>=0
False                                       # Interval for sin(π) contains 0
sage: sin(Ipi)>=x
False                                       # These intervals intersect
```

Defining U, we try to make it as small as possible keeping it locally positive around any vertrex (•).

3: How to check

$$S(\triangle) \geq U(\triangle) \tag{$\triangle$}$$

on each triangle △? (There is a continuum of them).

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3: How to check

$$S(\triangle) \geq U(\triangle) \qquad (\triangle)$$

on each triangle $\triangle$? (There is a continuum of them).

Interval arithmetic!

Delaunay triangulation properties → uniform bound on edge length:

Verify $S(\triangle_{e_1,e_2,e_3}) \geq U(\triangle_{e_1,e_2,e_3})$ where

$$e_1 = [r_a+r_b, r_a+r_b+2s] \; e_2 = [r_c+r_b, r_c+r_b+2s] \; e_3 = [r_a+r_c, r_a+r_c+2s]$$

Not precise enough (intervals intersect) → dichotomy

TODO

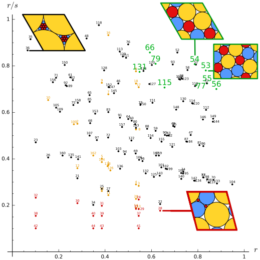
- Classify all the remaining cases

  modified Connelly's conjecture: triangulated is the densest among the packing using **all 3 sizes of discs**

- Find good lower bounds on the maximal density for other disc sizes (without triangulated packins)

  deformations of triangulated packings keep denisty high: flip and flow

  to fill the blanc space of the map



- Existence of a triangulated packing for a given set of discs – is it decidable?

  $\sim$ are there aperiodic triangulated packings?

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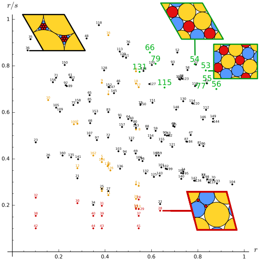

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Thank you for your attention!