A PROXIMAL BASED STRATEGY FOR SOLVING DISCRETE MUMFORD-SHAH AND AMBROSIO-TORTORELLI MODELS

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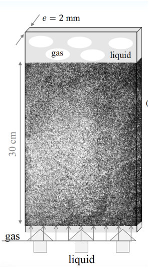
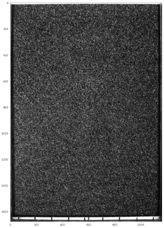
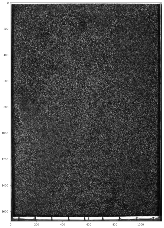
Background and motivation on edge detection



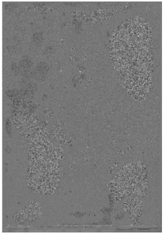
Multiphase flow of gas+liquid



at t_0 (liquid only)



at t_i (gas+liquid)



$\frac{(t_i)}{(t_0)}$

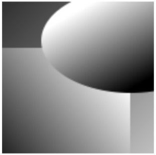
- Goal: estimate the interface between gas and liquid
- Datasize: large image (1677×1160 pixels), analysis need to be performed on a sequence of images.

Discrete Mumford-Shah model and related approaches

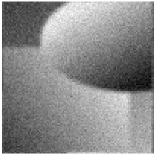
Degradation model:

$$\mathbf{z} = A\overline{\mathbf{u}} + n,$$

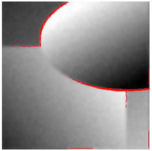
- $N = N_r N_c$: number of pixels
- $\mathbf{z} \in \mathbb{R}^M$: degraded image
- $\overline{\mathbf{u}} \in \mathbb{R}^N$: original image
- $A \in \mathbb{R}^{M \times N}$: linear degradation operator (e.g. a blur)
- $\mathbf{e} \in \mathbb{R}^{|\mathbb{E}|}$: edges detected, where $|\mathbb{E}| = N - N_r - N_c$
- $n \sim \mathcal{N}(0, \sigma^2 \mathbb{I}_M)$: additive white gaussian noise



$\overline{\mathbf{u}}$



$\mathbf{z}$



$\widehat{\mathbf{u}}$
$\widehat{\mathbf{e}}$ (colored in red on edges between pixels)

→ Goal: Jointly recover $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{e}}$.

Related approaches, DMS-like model

- Estimate $\widehat{\mathbf{e}}$ from $\widehat{\mathbf{u}}$:

$$\widehat{\mathbf{u}} \in \underset{\mathbf{u}\in\mathbb{R}^N}{\operatorname{Argmin}} \;\; \frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2 + \lambda\mathcal{P}(\mathbf{u}).$$

  - Thresholded-ROF based on TV denoising [Cai, Steidl, 2013]: $\mathcal{P}(\mathbf{u}) = \|D_0\mathbf{u}\|_{1,2}$
  - Blake-Zisserman model: $\mathcal{P}(\mathbf{u}) = \sum_i \min(|(D\mathbf{u})_i|^p, \alpha^p)$,
    [Strekalovskiy, Cremers, 2014] [Hohm et al., 2015]
  - Potts model: $\mathcal{P}(\mathbf{u}) = \|\nabla\mathbf{u}\|_0$.
    [Storath, Weinmann, 2015]

- Jointly estimate $\widehat{\mathbf{e}}$ and $\widehat{\mathbf{u}}$ using Mumford-shah model (discrete formulation):

  [Foare, Lachaud, Talbot, 2016] [Foare, Pustelnik, Condat, 2019]

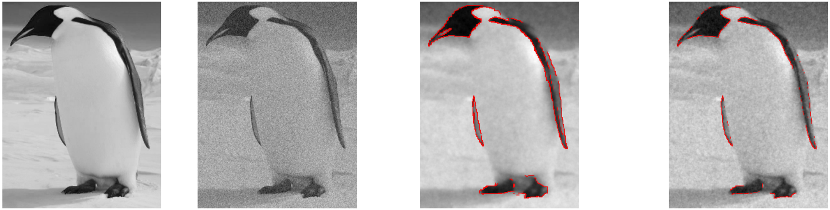
$$(\widehat{\mathbf{u}},\widehat{\mathbf{e}}) \in \operatorname{Argmin} \Phi(\mathbf{u},\mathbf{e}) := \frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2+\beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2+\lambda\mathcal{R}(\mathbf{e})$$

In the following we focus on this minimization problem

Proposed Discrete-Mumford-Shah model

$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \; \Phi(\mathbf{u},\mathbf{e}) := \frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2 + \beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2 + \lambda\mathcal{R}(\mathbf{e}).$$

- $\mathcal{R}$: edges penalization.
- $D_0 \in \mathbb{R}^{|\mathbb{E}|\times N}$: finite difference operator.
- $\mathbf{e} \in \mathbb{R}^{|\mathbb{E}|}$: edges between nodes whose value is 1 when a contour change is detected, and 0 otherwise.
- $\beta > 0$ and $\lambda > 0$: regularization parameters.



$\overline{\mathbf{u}}$ | $\mathbf{z}$ | $\widehat{\mathbf{u}}$, $\widehat{\mathbf{e}}$ $\beta=10, \lambda=0.01$ | $\widehat{\mathbf{u}}$, $\widehat{\mathbf{e}}$ $\beta=4, \lambda=0.01$

Proposed Discrete-Mumford-Shah model

$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \; \Phi(\mathbf{u},\mathbf{e}) := \frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2 + \beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2 + \lambda\mathcal{R}(\mathbf{e}),$$

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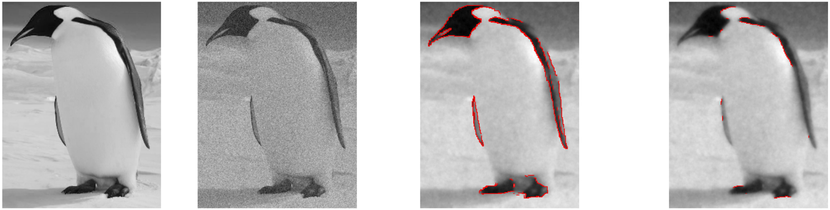


$\overline{\mathbf{u}}$ | $\mathbf{z}$ | $\widehat{\mathbf{u}}$, $\widehat{\mathbf{e}}$ $\beta=10, \lambda=0.01$ | $\widehat{\mathbf{u}}$, $\widehat{\mathbf{e}}$ $\beta=10, \lambda=0.05$

Choice of $\mathcal{R}(\mathbf{e})$

$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}}\ \Phi(\mathbf{u},\mathbf{e}) := \frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2 + \beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2 + \lambda\mathcal{R}(\mathbf{e}),$$

- $\mathcal{R}$: favors sparse solution on edges variable and convex.
  1. Ambrosio-Tortorelli approximation [Foare et al., 2016]:

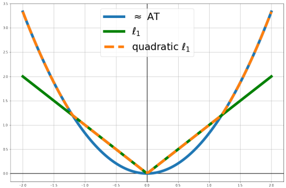
     $$\mathcal{R}(\mathbf{e}) = \frac{1}{4\varepsilon}\|\mathbf{e}\|_2^2 + \varepsilon\|D_1\mathbf{e}\|_2^2$$

  2. ℓ_1-norm: [Foare, Pustelnik, Condat, 2019] :

     $$\mathcal{R}(\mathbf{e}) = \|\mathbf{e}\|_1$$

  3. Quadratic-ℓ_1 [Foare, Pustelnik, Condat, 2019] :

     $$\mathcal{R}(\mathbf{e}) = \sum_{i=1}^{|\mathbb{E}|}\max\left\{|e_i|, \frac{e_i^2}{4\varepsilon}\right\}$$



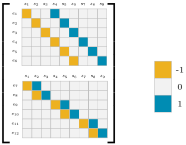
Design of D_0 and D_1

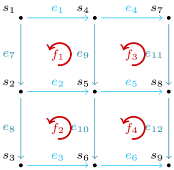
$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \ \Phi(\mathbf{u},\mathbf{e}) := \frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2 + \beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2 + \lambda\mathcal{R}(\mathbf{e}),$$

$$\text{with } \mathcal{R}(\mathbf{e}) = \frac{1}{4\varepsilon}\|\mathbf{e}\|_2^2 + \varepsilon\|D_1\mathbf{e}\|_2^2$$

- $D_0 : \mathbb{R}^N \to \mathbb{R}^{|\mathbb{E}|}$ maps $\mathbf{u}$ into a pair of $\mathbf{e} = (\mathbf{e}_h, \mathbf{e}_v)$ can be expressed by:

$$D_0 = \begin{bmatrix} H \\ V \end{bmatrix} =$$

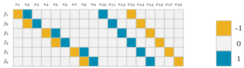




$(N = 9, |E| = 12)$

- $D_1 \in \mathbb{R}^{|\mathbb{E}|} \to \mathbb{R}^N$,

$$D_1 = \begin{bmatrix} V & -H \end{bmatrix} =$$



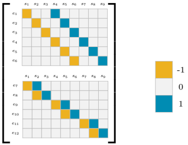
Design of D_0 and D_1

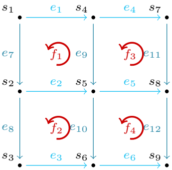
$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}}\ \Phi(\mathbf{u},\mathbf{e}) := \frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2 + \beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2 + \lambda\mathcal{R}(\mathbf{e}),$$

$$\text{with } \mathcal{R}(\mathbf{e}) = \frac{1}{4\varepsilon}\|\mathbf{e}\|_2^2 + \varepsilon\|D_1\mathbf{e}\|_2^2$$

- $D_0 : \mathbb{R}^N \to \mathbb{R}^{|\mathbb{E}|}$ maps $\mathbf{u}$ into a pair of $\mathbf{e} = (\mathbf{e}_h, \mathbf{e}_v)$ can be expressed by:

$$D_0 = \begin{bmatrix} H \\ V \end{bmatrix} =$$





$(N = 9, |E| = 12)$

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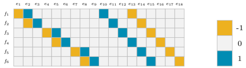
$$D_1 = \begin{bmatrix} V & -H \end{bmatrix} =$$



Proximal operator

Let $f : \mathcal{H} \to]-\infty, +\infty]$, convex, proper and lower semicontinuous, possibly non-smooth (i.e $\Gamma_0(\mathcal{H})$)

$$\widehat{\mathbf{x}} \in \operatorname*{Argmin}_{\mathbf{x} \in \mathcal{H}} f(\mathbf{x})$$

Proximal operator

Let $f : \mathcal{H} \to]-\infty, +\infty]$, convex, proper and lower semicontinuous, possibly non-smooth

$$\widehat{\mathbf{x}} \in \underset{\mathbf{x} \in \mathcal{H}}{\text{Argmin}} \; f(\mathbf{x})$$

Definition

$f \in \Gamma_0(\mathcal{H})$, when $\mathcal{H}$ models a real Hilbert space, the proximity operator of f is defined by, for some $\tau > 0$:

$$(\forall \mathbf{x} \in \mathcal{H}) \qquad \text{prox}_{\tau f}(\mathbf{x}) = \underset{\mathbf{y} \in \mathcal{H}}{\text{argmin}} \; f(\mathbf{y}) + \frac{1}{2\tau} \|\mathbf{y} - \mathbf{x}\|_2^2,$$

Proximal operator

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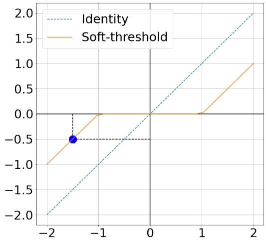
Example:

$f(x) = |x| \rightarrow \mathrm{prox}_{\tau f}(x) =$ soft-thresholding with a fixed threshold $\tau > 0$



Proximal algorithm: convex objective function

Let $f : \mathcal{H} \rightarrow]-\infty, +\infty]$, convex, proper and lower semicontinuous, possibly non-smooth

$$\widehat{\mathbf{x}} \in \underset{\mathbf{x} \in \mathcal{H}}{\operatorname{Argmin}} \; f(\mathbf{x})$$

Definition

$f \in \Gamma_0(\mathcal{H})$, when $\mathcal{H}$ models a real Hilbert space, the proximity operator of f for some $\tau > 0$ is defined by:

$$(\forall \mathbf{x} \in \mathcal{H}) \qquad \operatorname{prox}_{\tau f}(\mathbf{x}) = \underset{\mathbf{y} \in \mathcal{H}}{\operatorname{argmin}} \; f(\mathbf{y}) + \frac{1}{2\tau}\|\mathbf{y} - \mathbf{x}\|_2^2,$$

Proximal point iteration fo some $\tau > 0$:

$$\mathbf{x}^{[k+1]} = \operatorname{prox}_{\tau f}(\mathbf{x}^{[k]})$$

$\mathbf{x}^{[k]}$ converges to to a minimizer $\widehat{\mathbf{x}}$ of f [Bauschke, Combettes, 2011]

Difficulty: closed form of prox [cf. *proximity-operator.net*]

Proximal algorithm: convex objective function

f convex possibly non-smooth, g convex, differentiable and gradient L-Lipschitz

$$\widehat{\mathbf{x}} \in \underset{\mathbf{x} \in \mathcal{H}}{\operatorname{Argmin}}\ f(\mathbf{x}) + \mathrm{g}(\mathbf{x})$$

- Forward-backward splitting algorithm:

$$\mathbf{x}^{[k+1]} := \mathbf{prox}_{\tau f}(\mathbf{x}^{[k]} - \tau \nabla\ \mathrm{g}(\mathbf{x}^{[k]}))$$

If $0 < \tau < 2/L$, this scheme converges to a minimizer of $f + g$.

Gauss-Seidel iterations: Non-convex objective function

f, h are convex, possibly non-smooth, g is convex w.r.t each variable and differentiable,

$$\underset{\mathbf{u}\in\mathbb{R}^N, \mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \ \Phi(\mathbf{u}, \mathbf{e}) := f(\mathbf{u}, \mathbf{z}) + g(\mathbf{u}, \mathbf{e}) + h(\mathbf{e})$$

- Gauss-seidel scheme = alternating minimization:

Set $(\mathbf{u}^{[0]}, \mathbf{e}^{[0]}) \in (\mathbb{R}^N, \mathbb{R}^{|\mathbb{E}|})$
for $k \in \mathbb{N}$ **do**
  $\mathbf{u}^{[k+1]} \in \underset{\mathbf{u}}{\text{Argmin}}\ \Phi(\mathbf{u}, \mathbf{e}^{[k]})$
  $\mathbf{e}^{[k+1]} \in \underset{\mathbf{e}}{\text{Argmin}}\ \Phi(\mathbf{u}^{[k+1]}, \mathbf{e})$

PAM iterations (Proximal Alternating Minimization)

f, h are convex, possibly non-smooth, g is convex w.r.t each variable and differentiable,

$$\underset{\mathbf{u}\in\mathbb{R}^N, \mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \ \Phi(\mathbf{u},\mathbf{e}) := f(\mathbf{u},\mathbf{z}) + g(\mathbf{u},\mathbf{e}) + h(\mathbf{e})$$

Algorithm 1: PAM [Attouch et al. 2010]

Set $(\mathbf{u}^{[0]}, \mathbf{e}^{[0]}) \in (\mathbb{R}^N, \mathbb{R}^{|\mathbb{E}|})$

for $k \in \mathbb{N}$ **do**

$$\mathbf{u}^{[k+1]} = \underset{\mathbf{u}}{\text{Argmin}} \ \Phi(\mathbf{u}, \mathbf{e}^{[k]}) + \frac{c_k}{2}\|\mathbf{u} - \mathbf{u}^{[k]}\|^2$$

$$\mathbf{e}^{[k+1]} = \underset{\mathbf{e}}{\text{Argmin}} \ \Phi(\mathbf{u}^{[k+1]}, \mathbf{e}) + \frac{d_k}{2}\|\mathbf{e} - \mathbf{e}^{[k]}\|^2$$

Convergence guarantees to a critical point

PAM iterations (Proximal Alternating Minimization)

f, h are convex, possibly non-smooth, g is convex w.r.t each variable and differentiable,

$$\underset{\mathbf{u}\in\mathbb{R}^N, \mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \; \Phi(\mathbf{u},\mathbf{e}) := f(\mathbf{u},\mathbf{z}) + g(\mathbf{u},\mathbf{e}) + h(\mathbf{e})$$

Algorithm 2: PAM [Attouch et al. 2010]

Set $(\mathbf{u}^{[0]}, \mathbf{e}^{[0]}) \in (\mathbb{R}^N, \mathbb{R}^{|\mathbb{E}|})$

for $k \in \mathbb{N}$ **do**

$$\mathbf{u}^{[k+1]} = \text{prox}_{\frac{1}{c_k}\Phi(\cdot,\mathbf{e}^{[k]})}(\mathbf{u}^{[k]})$$

$$\mathbf{e}^{[k+1]} = \text{prox}_{\frac{1}{d_k}\Phi(\mathbf{u}^{[k+1]},\cdot)}(\mathbf{e}^{[k]})$$

PAM iterations (Proximal Alternating Minimization)

$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \; \Phi(\mathbf{u},\mathbf{e}) := f(\mathbf{u},\mathbf{z}) + g(\mathbf{u},\mathbf{e}) + h(\mathbf{e})$$

Algorithm 3: PAM [Attouch et al. 2010]

Set $(\mathbf{u}^{[0]}, \mathbf{e}^{[0]}) \in (\mathbb{R}^N, \mathbb{R}^{|\mathbb{E}|})$

for $k \in \mathbb{N}$ **do**

$$\mathbf{u}^{[k+1]} = \text{prox}_{\frac{1}{c_k} f(\cdot;z) + \frac{1}{c_k} g(\cdot,\mathbf{e}^{[k]})}(\mathbf{u}^{[k]})$$

$$\mathbf{e}^{[k+1]} = \text{prox}_{\frac{1}{d_k} g(\mathbf{u}^{[k+1]},\cdot) + \frac{1}{d_k} h}(\mathbf{e}^{[k]})$$

Difficulty: Closed form expression for the prox of the sum of 2 functions.

$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \; \Phi(\mathbf{u},\mathbf{e}) := \underbrace{\frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2}_{f(\mathbf{u},\mathbf{z})} + \underbrace{\beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2}_{g(\mathbf{u},\mathbf{e})} + \underbrace{\lambda\mathcal{R}(\mathbf{e})}_{h(\mathbf{e})}$$

PALM iterations (Proximal Linearized Alternating Minimization)

$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \ \Phi(\mathbf{u},\mathbf{e}) := f(\mathbf{u},\mathbf{z}) + g(\mathbf{u},\mathbf{e}) + h(\mathbf{e})$$

Algorithm 4: PALM [Bolte, Sabach, Teboulle, 2013]

Initialization: $\mathbf{u}^{[0]} \in \mathbb{R}^N, \mathbf{e}^{[0]} \in \mathbb{R}^{|\mathbb{E}|}$

while $\frac{\Phi^{[k+1]}-\Phi^{[k]}}{\Phi^{[k]}} < 1\text{e}{-6}$ *and* $k \in \mathbb{N}$ **do**

- Set $\gamma > 1$ and $c_k = \gamma L_1(\mathbf{e}^{[k]})$;
- $\mathbf{u}^{[k+1]} = \text{prox}_{\frac{1}{2c_k}f(\cdot,\mathbf{z})}(\mathbf{u}^{[k]} - \frac{1}{c_k}\nabla_{\mathbf{u}} g(\mathbf{u}^{[k]},\mathbf{e}^{[k]}))$;
- Set $\eta > 1$ and $d_k = \eta L_2(\mathbf{u}^{[k+1]})$;
- $\mathbf{e}^{[k+1]} = \text{prox}_{\frac{1}{d_k}h}(\mathbf{e}^{[k]} - \frac{1}{d_k}\nabla_{\mathbf{e}} g(\mathbf{u}^{[k+1]},\mathbf{e}^{[k]}))$

Difficulty: closed form of prox in the update of the edge variable.

PALM iterations (Proximal Linearized Alternating Minimization)

The update of the edge variable in Algorithm 2 takes a closed form that is:

Proposition

- $\mathcal{R}(\mathbf{e}) = \|\mathbf{e}\|_1$:

$$\mathbf{e}^{[k+1]} = soft_{\frac{\lambda}{d_k}}(\mathbf{e}^{[k]})$$

- $\mathcal{R}(\mathbf{e}) = \frac{1}{4\varepsilon}\|\mathbf{e}\|_2^2 + \varepsilon\|D_1\mathbf{e}\|_2^2$:

$$\mathbf{e}^{[k+1]} = \left[\frac{2\lambda\varepsilon}{d_k} {D_1}^* D_1 + \left(1 + \frac{2\lambda}{4\varepsilon d_k}\right)\mathrm{Id}\right]^{-1} \mathbf{e}^{[k]}.$$

  - *Need to modify D_1 into periodic-D_1 to favor the computation of inverse operator above.*
  - *Efficiently inversion in Fourier if D_1 is circulant*

SL-PAM iterations

$$\underset{\mathbf{u}\in\mathbb{R}^N,\mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}{\text{minimize}} \; \Phi(\mathbf{u},\mathbf{e}) := \underbrace{\frac{1}{2}\|A\mathbf{u}-\mathbf{z}\|_2^2}_{f(\mathbf{u},\mathbf{z})} + \underbrace{\beta\|(1-\mathbf{e})\odot D_0\mathbf{u}\|_2^2}_{g(\mathbf{u},\mathbf{e})} + \underbrace{\lambda\mathcal{R}(\mathbf{e})}_{h(\mathbf{e})}$$

Algorithm 5: SL-PAM [Foare, Pustelnik, Condat, 2017]

Initialization: $\mathbf{u}^{[0]} \in \mathbb{R}^N, \mathbf{e}^{[0]} \in \mathbb{R}^{|\mathbb{E}|}$

while $\frac{\Phi^{[k+1]}-\Phi^{[k]}}{\Phi^{[k]}} < 1\mathrm{e}{-6}$ *and* $k \in \mathbb{N}$ **do**

- Set $\gamma > 1$ and $c_k = \gamma L_1(\mathbf{e}^{[k]})$
- $\mathbf{u}^{[k+1]} = \mathrm{prox}_{\frac{1}{c_k}f(\cdot,\mathbf{z})}\left(\mathbf{u}^{[k]} - \frac{1}{c_k}\nabla_{\mathbf{u}} g(\mathbf{e}^{[k]},\mathbf{u}^{[k]})\right)$
- Set $d_k > 0$
- $\mathbf{e}^{[k+1]} = \mathrm{prox}_{\frac{1}{d_k}(g(\cdot,\mathbf{u}^{[k+1]})+h)}\left(\mathbf{e}^{[k]}\right)$

→ Difficulty: computation of $\mathrm{prox}_{\frac{1}{d_k}(\beta\mathcal{S}(\cdot,\mathbf{u}^{[k+1]})+\lambda\mathcal{R})}\left(\mathbf{e}^{[k]}\right)$

SL-PAM iterations

→ Difficulty: computation of $\text{prox}_{\frac{1}{d_k}(\beta\mathcal{S}(\cdot,\mathbf{u}^{[k+1]})+\lambda\mathcal{R})}\left(\mathbf{e}^{[k]}\right)$

Proposition (Foare, Pustelnek, Condat, 2019)

We assume that $\mathcal{S}$ is separable, i.e:

$$(\forall \mathbf{e} = (\mathbf{e}_i)_{1\leq i\leq|\mathbb{E}|}) \qquad \mathcal{R}(\mathbf{e}) = \sum_{i=1}^{|\mathbb{E}|} \sigma_i(\mathbf{e}_i),$$

- $\sigma_i : \mathcal{R}^{|\mathbb{E}|} \to]-\infty, +\infty]$ *and whose proximity operator has a closed form expression*
- *Let $d_k > 0$, then at iteration $k \in \mathbb{N}$, the updating step on $\mathbf{e}^{[k+1]}$ in Algorithm 3 is:*

$$\mathbf{e}^{[k+1]} = \left(prox_{\frac{\lambda\sigma_i}{2\beta(D_0\mathbf{u}^{[k+1]})_i^2+d_k}} \frac{\beta(D_0\mathbf{u}^{[k+1]})_i^2 + \frac{d_k \mathbf{e}_i^{[k]}}{2}}{\beta(D_0\mathbf{u}^{[k+1]})_i^2 + \frac{d_k}{2}} \right)_{i\in\mathbb{E}}$$

SL-PAM iterations

Proposition (Le, Foare, Nelly, 2021)

When $\mathcal{R}(\mathbf{e}) = \frac{1}{4\varepsilon}\|\mathbf{e}\|_2^2 + \varepsilon\|D_1\mathbf{e}\|_2^2$, then the update of the edge variable in Algorithm 3 takes a closed form that is:

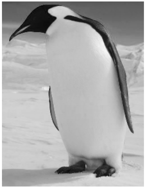
$$\mathbf{e}^{[k+1]} = \left[\frac{2\lambda\varepsilon}{d_k}D_1^*D_1 + \frac{2\beta}{d_k}\mathrm{diag}\left((D_0\mathbf{u}^{[k+1]})^2\right) + \dots \right.$$

$$\left. \left(\frac{\lambda}{2\varepsilon d_k} + 1\right)\mathrm{Id}\right]^{-1}\left(\frac{2\beta}{d_k}(D_0\mathbf{u}^{[k+1]})^2 + \mathbf{e}^{[k+1]}\right).$$

Difficulty: $D_1^*D_1$ and $\mathrm{diag}(D_0\mathbf{u}^{[k]})$ not-diagonalisable in the same basis.

Experiments

$N = 572 \times 558,\ \beta = 10,\ \lambda = 0.01$



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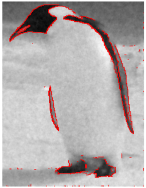
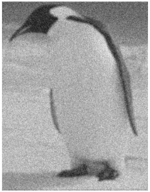


SL-PAM ℓ_1



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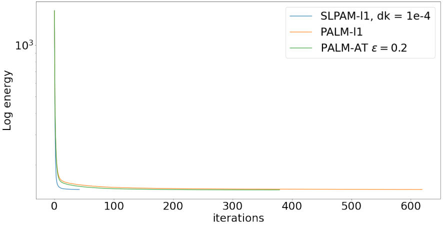
PALM-ℓ_1

PALM-AT$_{\varepsilon=0.2}$

Which scheme gives the best result?

Experiments

$N = 572 \times 558$, $\beta = 10$, $\lambda = 0.01$

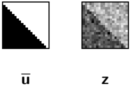


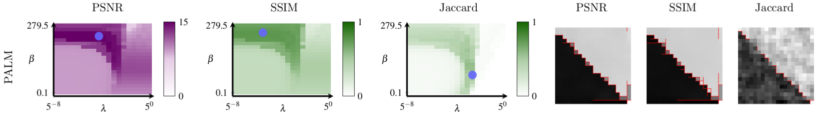
	CT	Number of iterations
SLPAM-l1	10s	43
PALM-l1	4min 01s	620
SLPAM-AT eps=0.2	NA	NA
PALM-AT eps=0.2	2 mins 27 s	380

Table: Computation time and number of iterations

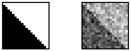
SL-PAM ℓ_1 is the fastest scheme. However, which scheme gives the best result in terms of contour detection?

Experiments





Experiments



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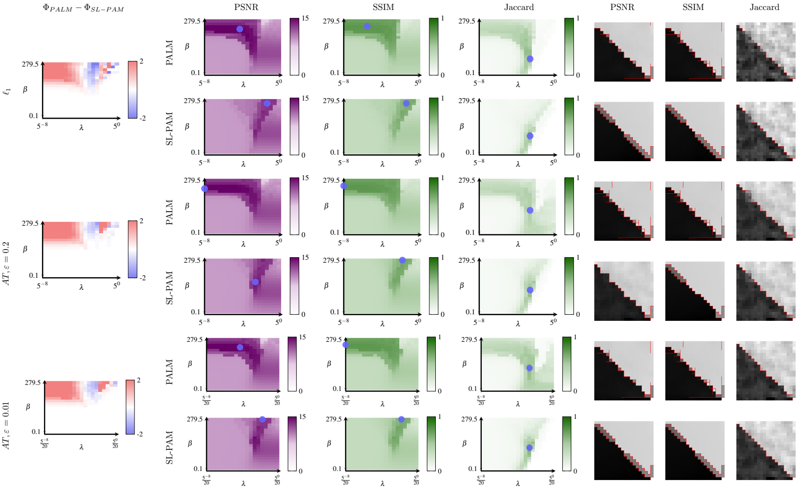


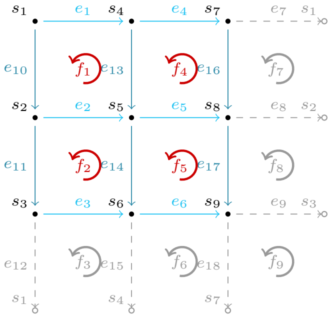
Conclusions and future works

- Discrete-Mumford-Shah and Ambrosio-Tortorelli models
- SL-PAM and PALM for solving both models
- SL-PAM fastest for ℓ_1 but not applicable for AT for large data set
- SL-PAM AT, SL-PAM-l1, PALM-AT, PALM-l1 very close performance in terms of contour detection
- Importance of β and λ choice

- Deeper convergence analysis of PALM and SL-PAM
- Application to multiphasic flow experiment leading to modifications in the data-term.

THANK YOU FOR YOUR ATTENTION!!!

PALM iterations



$H_{per} =$

$V_{per} =$