Discrete Vector Bundles with Connection and the Bianchi Identity

Path from DEC to discrete differential geometry?

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Outline

Main ideas and results

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Discrete theory

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Section 1

Main ideas and results

Vector bundles and differential geometry

Ingredients of smooth theory

- E a *vector bundle* over a smooth manifold M

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- Vector spaces of E-*valued differential* k-*forms* $\Lambda^k(M; E)$

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- 0-forms are also called *sections* $\Gamma(E) = \Lambda^0(M; E)$

Vector bundles and differential geometry

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- Vector spaces of E-*valued differential* k-*forms* $\Lambda^k(M; E)$
- 0-forms are also called *sections* $\Gamma(E) = \Lambda^0(M; E)$
- A *connection* $\nabla \colon \Gamma(E) \to \Lambda^1(M; E)$

Vector bundles and differential geometry

Connection ∇ leads to exterior covariant derivative d_∇

- Connection allows one to connect different fibers via parallel transport (hence the name)

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- This leads to a sequence

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- Unlike de Rham complex, d^2 need not be 0

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- Curvature F is an endomorphism valued 2-form

Vector bundles and differential geometry

How things fit together

- $\Lambda^\bullet(M)$ of differential forms acts on $\Lambda^\bullet(M;E)$ through linear maps

$$\Lambda^k(M) \times \Lambda^l(M;E) \to \Lambda^{k+l}(M;E), \qquad (w,\alpha) \mapsto w \wedge \alpha \tag{1}$$

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- d_∇ commuting with pullbacks is a further generalization

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- Other consequences are about trivializability and reduction of structure group

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Theorem (Structure-preserving discretization)

There exist maps

$$\nabla : \Gamma(E) = C^0(X; E) \to C^1(X; E)$$
$$d_\nabla : C^k(X; E) \to C^{k+1}(X; E)$$
$$\wedge : C^k(X) \times C^l(X; E) \to C^{k+l}(X; E)$$
$$C^k(X; \mathrm{Hom}(E)) \times C^l(X; E) \to C^{k+l}(X; E)$$
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and $F \in C^2(X; \mathrm{Hom}(E))$ *such that:*

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(vi) $d_\nabla F = 0$ *(Bianchi identity)*

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Trivializability and reduction of structure group

Definition

A discrete vector bundle with connection is *flat* (or the connection is flat) if the parallel transport between any two points only depends on the simple homotopy class of the path connecting the two points.

- Sometimes the parallel transport maps can be simplified via bundle isomorphism

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- Not covered in this talk

Section 2

Discrete theory

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U_{ji} can be obtained by solving parallel transport ODEs

Vector bundle valued cochains

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A *vector bundle valued* k-*cochain* α assigns to each k-simplex σ of X an element $\langle \alpha, \sigma \rangle^l$ of E_l where l is a vertex in the simplex. The vector space of k-cochains is denoted $C^k(X; E)$. A *section* s is a vector bundle valued 0-cochain, i.e., a vector $s_i \in E_i$ for each vertex.

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- For permuted simplex $[v_{\tau(0)}, \ldots, v_{\tau(k)}]$

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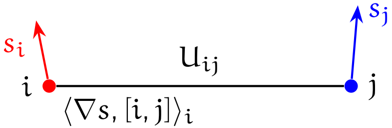
- $\Lambda^k(M; E)$ discretized to $C^k(X; E)$ using local trivializations

Discrete covariant derivative ∇

Definition

The *discrete covariant derivative* or *connection* is a map $\nabla\colon C^0(X,E) \to C^1(X,E)$ which to a section s assigns the vector-valued 1-cochain defined by its value on edges $[i,j]$ by

$$\langle \nabla s, [i,j] \rangle_i := U_{ij} s_j - s_i \,. \tag{3}$$



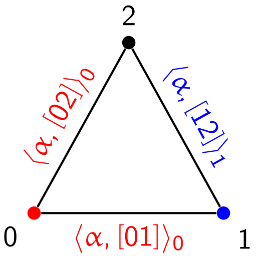
Discrete exterior covariant derivative d_∇

Definition

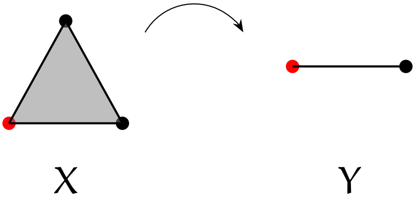
Given $\alpha \in C^{k-1}(X, E)$ and $[0 \dots k]$ a k-simplex, the discrete *exterior covariant derivative* d_∇ is defined by

$$\langle d_\nabla \alpha, [0 \dots k] \rangle_0 := U_{01} \langle \alpha, [1 \dots k] \rangle_1 + \sum_{j=1}^{k} (-1)^j \langle \alpha, [0 \dots \hat{j} \dots k] \rangle_0 \,.$$

[Kock, 1996]

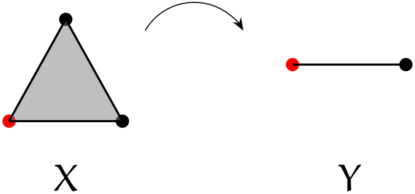


Abstract simplicial maps

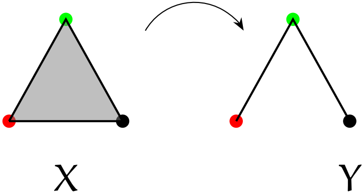


Abstract simplicial map

Abstract simplicial maps

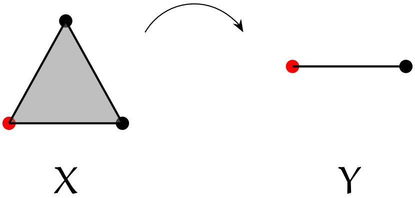


Abstract simplicial map

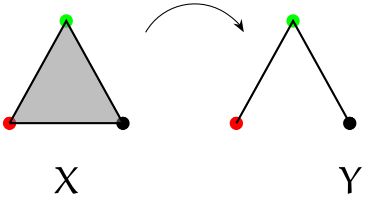


Not an abstract simplicial map

Abstract simplicial maps



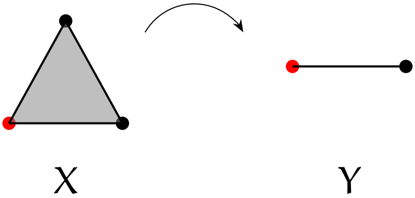
Abstract simplicial map



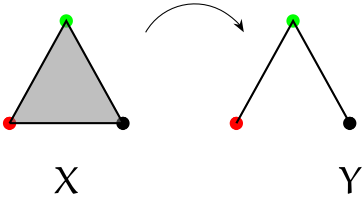
Not an abstract simplicial map

- Collapse is OK

Abstract simplicial maps



Abstract simplicial map

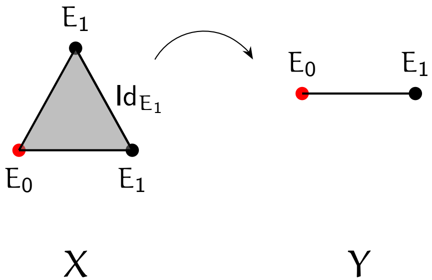


Not an abstract simplicial map

- Collapse is OK
- Vertices that were "near" (edge connected) remain "near"

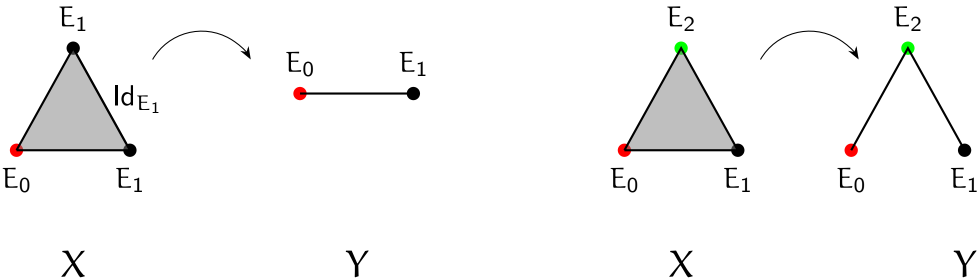
Abstract simplicial maps

Discrete pullback bundle



Abstract simplicial maps

Discrete pullback bundle



Discrete wedge product

Definition

Given $\alpha \in C^k(X, E)$ and $w \in C^l(X)$ their *wedge product* is defined by

$$\langle \alpha \wedge w, [0 \ldots k+l] \rangle_0 = \frac{1}{k+l+1!} \sum_{\tau \in S_{k+l+1}} \operatorname{sgn}(\tau) \langle \alpha \smile w, [\tau(0), \tau(1) \ldots \tau(k+l)] \rangle_0$$

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- Transports that are needed before adding vectors are hidden in the notation

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- Generalization of the DEC wedge product
- Transports that are needed before adding vectors are hidden in the notation
- Anti-commutative but not associative (same as DEC)

Homomorphism valued cochains

Definition

A *homomorphism-valued* k-cochain A is a map whose value at each k-simplex $[0 \ldots k]$ is a linear map $E_k \to E_0$. The bundle of *homomorphism-valued* k*-cochains* is denoted $C^k(X; \mathrm{Hom}(E))$. Given $A \in C^k(X; \mathrm{Hom}(E)$ and $\alpha \in C^l(X; E)$ the action of A on α is defined as:

$$\langle A\, \alpha, [0 \ldots k+l] \rangle = \langle A, [0 \ldots k] \rangle \langle \alpha, [k \ldots k+l] \rangle \ .$$

d_∇ for homomorphism valued cochains

Definition

Let $A \in C^k(X; \mathrm{Hom}(E))$. Then $d_\nabla A$ is definied by its evaluaton on a simplex $[0 \dots k+1]$ by

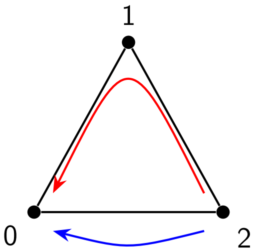
$$\langle d_\nabla A, [0 \dots k+1] \rangle_0 := U_{01} \langle A, [1 \dots k+1] \rangle_1 + \sum_{j=1}^{k} \left[(-1)^j \langle A, [0 \dots \hat{j} \dots k+1] \rangle_0 \right] + (-1)^{k+1} \langle A, [0 \dots k] \rangle U_{k,(k+1)} .$$

Discrete curvature 2-cochain

Definition

The *discrete curvature* is a homomorphism-valued 2-cochain, $F \in C^2(X; \mathrm{Hom}(E))$, defined on a triangle $[012]$ by

$$\langle F, [012] \rangle = U_{01} U_{12} - U_{02} \,.$$



Same information as “holonomy minus identity” but fits with structure-preservation

Section 3

Summary and outlook

Summary of main results

Structure-preserving discretization

(i) For $f \in C^0(X)$, section $s \in C^0(X, E)$: $\nabla(f \wedge s) = df \wedge s + f \wedge \nabla s$

(ii) For $\alpha \in C^k(X, E)$ and $w \in C^l(X)$: $d_\nabla(\alpha \wedge w) = d_\nabla \alpha \wedge w + (-1)^k \alpha \wedge dw$

(iii) Given abstract simplicial map $\varphi : X' \to X$: $\varphi^*(\alpha \wedge w) = \varphi^* \alpha \wedge \varphi^* w$

(iv) $\varphi^* d_\nabla = d_\nabla \varphi^*$

(v) $d_\nabla d_\nabla \alpha = F\, \alpha$

(vi) $d_\nabla F = 0$ (Bianchi identity)

Trivializability and reduction of structure group

(i) Characterization of trivializabilty in terms of flatness

(ii) Existence of trivial subbundles in terms of flat sections

Conclusions and outlook

- A combinatorial discretization of vector bundles with connection has been built

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Conclusions and outlook

- A combinatorial discretization of vector bundles with connection has been built
- Using d_∇ as a building block curvature emerges from the discretization
- Bundle metric has been studied, but not Riemannian metric
- Not clear how to recognize, for example, the tangent bundle
- There are other ways to organize the discrete connection