Numerical resolution of semi-discrete Generated Jacobian equations

Anatole Gallouet
Supervisors: Quentin Mérigot and Boris Thibert

Laboratoire Jean Kuntzmann, Grenoble

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Nonimaging optics

Transfer of light from a source to a target.

INPUT:

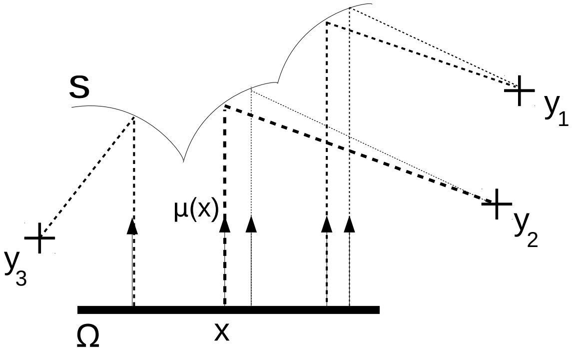
- Light source, measure μ.
- Destination target, measure ν.

OUTPUT:

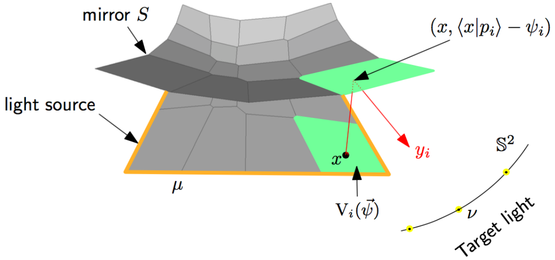
- A mirror surface S reflecting μ on ν.

Semi-discrete setup

- Source $\Omega \subset \mathbb{R}^d$, with intensity $\mu(E) = \int_E \rho(x)dx$
- Target $Y = (y_i)_{1 \leq i \leq N}$, with intensity $\nu = \sum_i \nu_i \delta_{y_i}$.
- Mass balance: $\mu(\Omega) = \nu(Y)$.



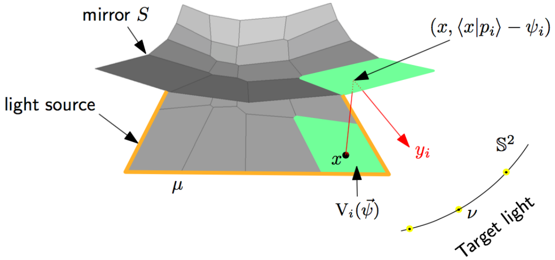
Far field reflector problem



We choose S to be a maximum of planes, so it is the graph of

$$u : x \to \max_{1 \leq i \leq N} \langle x, p_i \rangle - \psi_i.$$

Far field reflector problem

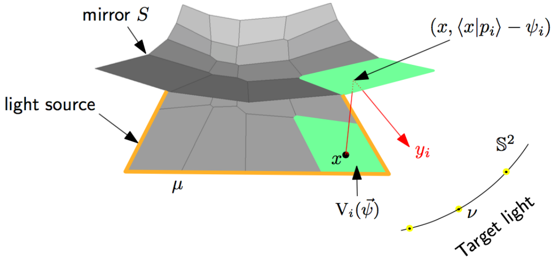


$$V_i(\psi) = \{x \in \Omega | \forall j : \langle x, p_i\rangle - \psi_i \geq \langle x, p_j\rangle - \psi_j\}$$

Far field reflector problem:

Find $\psi = (\psi_i)_{1\leq i\leq N}$ s.t. $\forall i : \mu(V_i(\psi)) = \nu_i$

Far field reflector problem



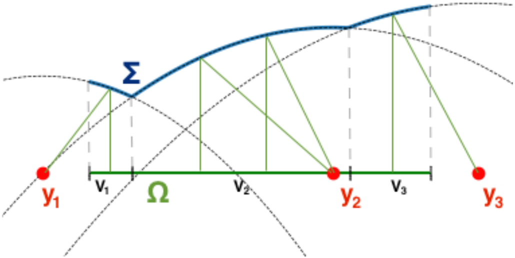
$$V_i(\psi) = \{x \in \Omega | \forall j : \langle x, p_i \rangle - \psi_i \geq \langle x, p_j \rangle - \psi_j\}$$

Far field reflector problem:

$$\text{Find } \psi = (\psi_i)_{1 \leq i \leq N} \text{ s.t. } \forall i : \mu(V_i(\psi)) = \nu_i$$

Linear in $\psi \rightarrow$ Optimal transport

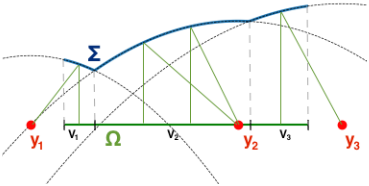
Near field reflector problem



Here, Σ is a maximum of paraboloids of focus y_i.

$$u(x) = \max_{1 \leq i \leq N} \frac{1}{2\psi_i} - \frac{\psi_i}{2} ||x - y_i||^2$$

Near field reflector problem

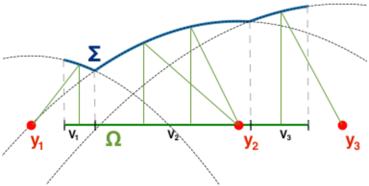


$$V_i(\psi) = \left\{ x \in \Omega | \forall j : \frac{1}{2\psi_i} - \frac{\psi_i}{2} ||x - y_i||^2 \geq \frac{1}{2\psi_j} - \frac{\psi_j}{2} ||x - y_j||^2 \right\}$$

Near field reflector problem:

Find $\psi = (\psi_i)_{1 \leq i \leq N}$ s.t. $\forall i \in [\![1, N]\!] : \mu(V_i(\psi)) = \nu_i$

Near field reflector problem



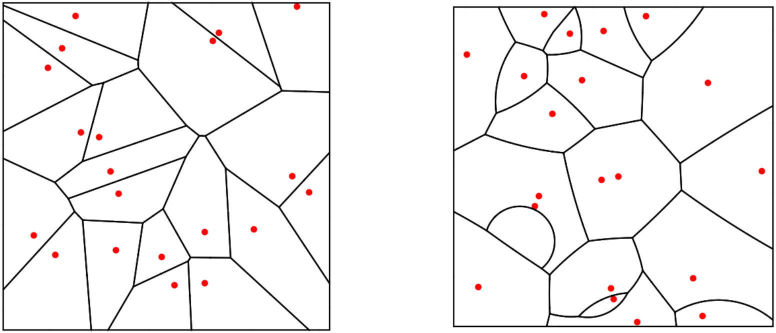
$$V_i(\psi) = \left\{ x \in \Omega | \forall j : \frac{1}{2\psi_i} - \frac{\psi_i}{2}||x - y_i||^2 \geq \frac{1}{2\psi_j} - \frac{\psi_j}{2}||x - y_j||^2 \right\}$$

Near field reflector problem:

Find $\psi = (\psi_i)_{1 \leq i \leq N}$ s.t. $\forall i \in [\![1, N]\!] : \mu(V_i(\psi)) = \nu_i$

Not linear in $\psi \rightarrow$ Not optimal transport

Comparison of the diagrams



(a) $(V_i)_{1\leq i\leq N}$ in the Far field case. (b) $(V_i)_{1\leq i\leq N}$ in the Near field case.

Figure 1: Comparison of Power and Mobius Diagram

Generating function

Definition (Generating function)

A function $G : \Omega \times Y \times \mathbb{R} \to \mathbb{R}$ is called a generating function if it satisfies (Reg), (Mono), (Twist) and (UC).

Definition (Generalized Laguerre cells)

We define the generalized Laguerre cells associated to a generating function G for $i \in [\![1, N]\!]$ by

$$\mathrm{Lag}_i(\psi) = \{x \in \Omega | \forall j \in [\![1, N]\!], G(x, y_i, \psi_i) \geq G(x, y_j, \psi_j)\}$$

Far field parallel reflector:

$$G(x, y, v) = \langle x, p \rangle - v$$

Near field parallel reflector:

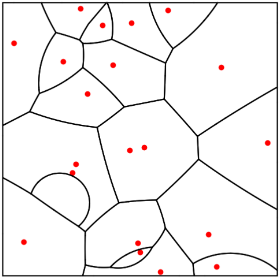
$$G(x, y, v) = \frac{1}{2v} - \frac{v}{2}||x - y||^2$$

Semi-discrete Generated Jacobian equation (Trudinger, 14)

The generated Jacobian equation consists in finding $\psi \in \mathbb{R}^N$ such that

$$H(\psi) = \nu \qquad \text{(GJE)}$$

where the function H is given by $H(\psi) = (\mu(\text{Lag}_i(\psi)))_{1 \leq i \leq N}$.



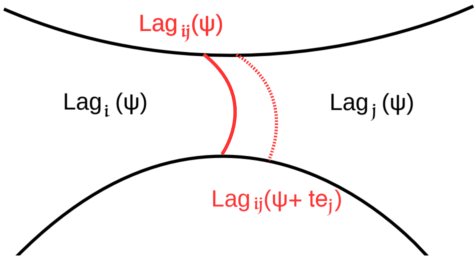
Generalizes semi-discrete O.T. problems (in the dual form).

Differential of H

Proposition

Under an hypothesis of genericity of Y, H is of class $\mathcal{C}^1$ and for $i \neq j$:

$$\begin{cases} \dfrac{\partial H_j}{\partial \psi_i}(\psi) = \displaystyle\int_{\mathrm{Lag}_{ij}(\psi)} \rho(x) \frac{|G_v(x, y_i, \psi_i)|}{||G_x(x, y_j, \psi_j) - G_x(x, y_i, \psi_i)||} d\mathcal{H}^{d-1}(x) \geq 0 \\ \dfrac{\partial H_i}{\partial \psi_i}(\psi) = -\displaystyle\sum_{j \neq i} \frac{\partial H_j}{\partial \psi_i}(\psi) \end{cases}$$



Properties of DH

$$\mathcal{S}^+ = \left\{\psi \in \mathbb{R}^N | \forall i, H_i(\psi) > 0\right\}$$

Proposition

- *$DH(\psi)$ the differential of H is of rank $N-1$ on $\mathcal{S}^+$.*
- *The image of DH is* $\operatorname{im}(DH(\psi)) = \mathbb{1}^\perp$ *where* $\mathbb{1} = (1, \cdots, 1) \in \mathbb{R}^N$.
- $\ker(DH(\psi)) = \operatorname{span}(w)$ *with* $w_i > 0$.

Proposition (Unique descent direction)

Let $\psi \in \mathcal{S}^+$, then the system:

$$\begin{cases} DH(\psi)u = H(\psi) - \nu \\ u_1 = 0 \end{cases} \tag{1}$$

has a unique solution.

Newton algorithm to solve Generated Jacobian Equations

$$\mathcal{S}^\delta = \left\{\psi \in \{\alpha\} \times [\beta,\gamma]^{(N-1)} | \forall i \in [\![1,N]\!], H_i(\psi) \geq \delta\right\}$$

Require: $\psi^0 \in \mathcal{S}^\delta$ and precision ϵ

Ensure: ψ such that $||H(\psi) - \nu|| \leq \epsilon$

1: $k \leftarrow 0$

2: **while** $||H(\psi^k) - \nu|| > \epsilon$ **do**

3: Compute the descent direction u^k solution of (1)

4: Let $\psi^{k,\tau} = \psi^k - \tau u^k$, we compute

$$\tau^k = \max\left\{\tau \in 2^{-\mathbb{N}}, ||H(\psi^{k,\tau}) - \nu|| \leq (1 - \frac{\tau}{2})||H(\psi^k) - \nu||\right\}$$

under the condition $\psi^{k,\tau} \in \mathcal{S}^\delta$.

5: $\psi^{k+1} \leftarrow \psi^k - \tau^k u^k$ and $k \leftarrow k+1$

6: **end while**

7: **return** ψ^k

Convergence of the algorithm

Theorem

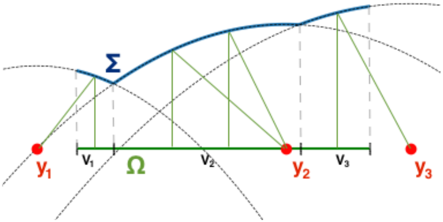
If Ω is a connected compact set, and under some assumptions on Y. If we choose $2\delta \leq \min_{1 \leq i \leq N} \nu_i$, then the algorithm converges in a finite number of steps.

Sketch of proof

$K^\delta = \{\psi \in S^\delta, ||H(\psi) - \nu|| \leq ||H(\psi^0) - \nu||\}$ is a non empty compact set.
At any iteration, we have $\psi^k \in K^\delta$.
By compactness, for any $k \in \mathbb{N}$, $\tau^k \geq \tau_{min}$ which gives:

$$||H(\psi^k) - \nu)|| \leq \left(1 - \frac{\tau_{min}}{2}\right)^k ||H(\psi^0) - \nu||$$

Implementation for the near field reflector



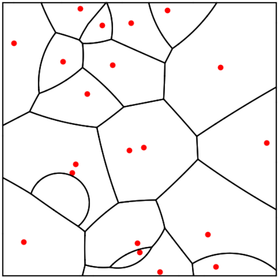


Figure 2: Near field reflector problem

Computing the diagram

Möbius diagram

$$V_i = \{x \in \Omega | \forall j \in [\![1, N]\!] : \lambda_i ||x - p_i||^2 - \mu_i \leq \lambda_j ||x - p_j||^2 - \mu_j\}$$

Power diagram

$$Pow_i = \{x \in \Omega | \forall j \in [\![1, N]\!] : ||x - c_i||^2 - r_i \leq ||x - c_j||^2 - r_j\}$$

Lemma (Boissonnat,Wormser,Yvinec,07)

$$V_i = \Pi(Pow_i \cap P)$$

with $V_i \subset \mathbb{R}^n \times \{0\}$, $Pow_i \subset \mathbb{R}^{n+1}$ *and* $P = \{(x, ||x||^2) | x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$.
Π *is the orthogonal projection of* $\mathbb{R}^{n+1}$ *on* $\mathbb{R}^n \times \{0\}$.

Newton algorithm for 5000 points

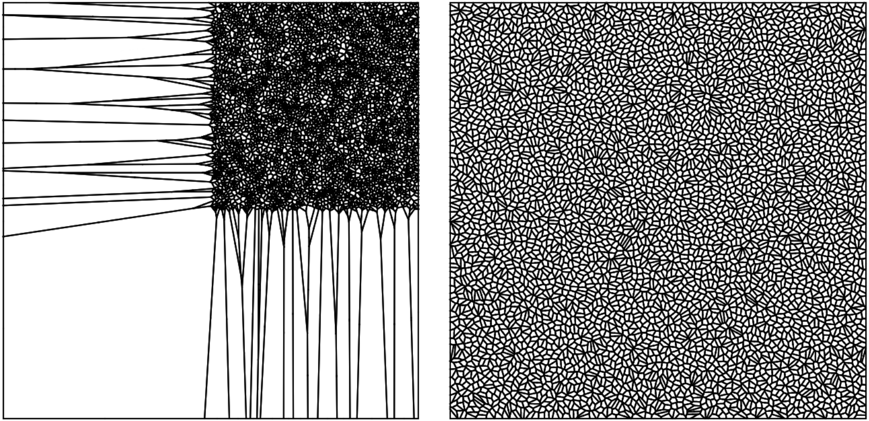
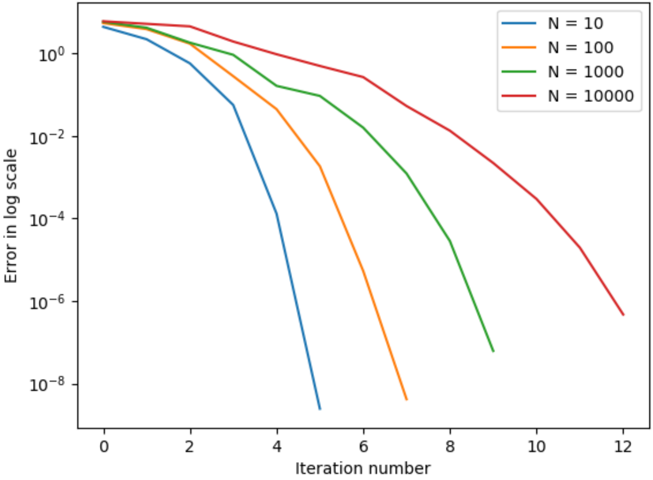


Figure 3: Initial and final diagram for 5000 points
$\Omega = [-1, 1]^2$ and $Y \subset [0, 1]^2$

Convergence rate



Conclusion

Contribution

- Adaptation of an algorithm for O.T. to generated Jacobian equations.
- Proof of convergence
- Implementation for the Near Field reflector.

Perspectives

- Uniqueness to (GJE) (semi-discrete case).
- initilization of the algorithm.

Genericity of Y

Definition

(Genericity of Y). For i, j, k three distinct indices in $[\![1, N]\!]$, we define $G_{ij}(\psi) = \{x \in \Omega | G(x, y_i, \psi_i) = G(x, y_j, \psi_j)\}$ and $G_{ijk}(\psi) = G_{ij}(\psi) \cap G_{ik}(\psi)$.

- We say that Y is generic with respect to G if for all distinct indices i, j, k and $\psi \in \mathbb{R}^N$ we have

$$\mathcal{H}^{d-1}(G_{ijk}(\psi)) = 0$$

.

- We say that Y is generic with respect to X if for all distinct indices i, j and $\psi \in \mathbb{R}^N$ we have

$$\mathcal{H}^{d-1}(G_{ij}(\psi) \cap \partial X) = 0$$

Conditions on the Generating function

- *The regularity condition:* $(x, y, v) \mapsto G(x, y, v)$ is continuously differentiable in x and v, and

$$\forall \alpha \in \mathbb{R}, \sup_{(x,y,v)\in\Omega\times Y\times]-\infty,\alpha]} |G_x(x, y, v)| < +\infty \qquad \text{(Reg)}$$

- *The monotonicity condition:*

$$\forall (x, y, v) \in \Omega \times Y \times \mathbb{R} : G_v(x, y, v) < 0 \qquad \text{(Mono)}$$

- *The twist condition:*

$$(y, v) \mapsto (G(x, y, v), G_x(x, y, v)) \text{ is injective for any } x \in X \qquad \text{(Twist)}$$

- *The uniform convergence condition:*

$$\forall y \in Y, \lim_{v\to-\infty} \inf_{x\in\Omega} G(x, y, v) = +\infty \qquad \text{(UC)}$$