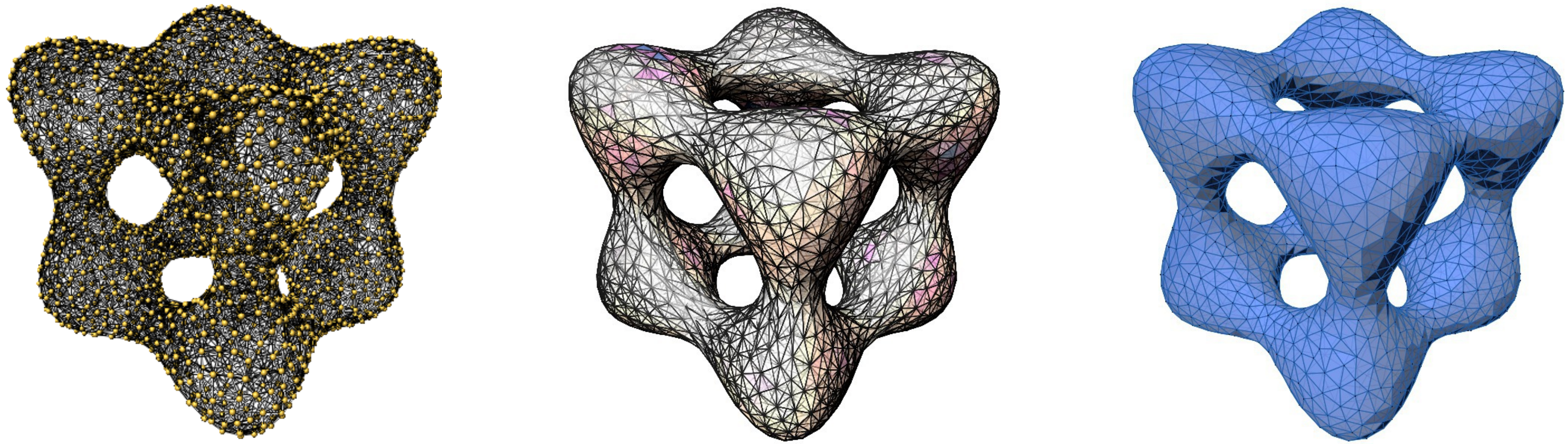
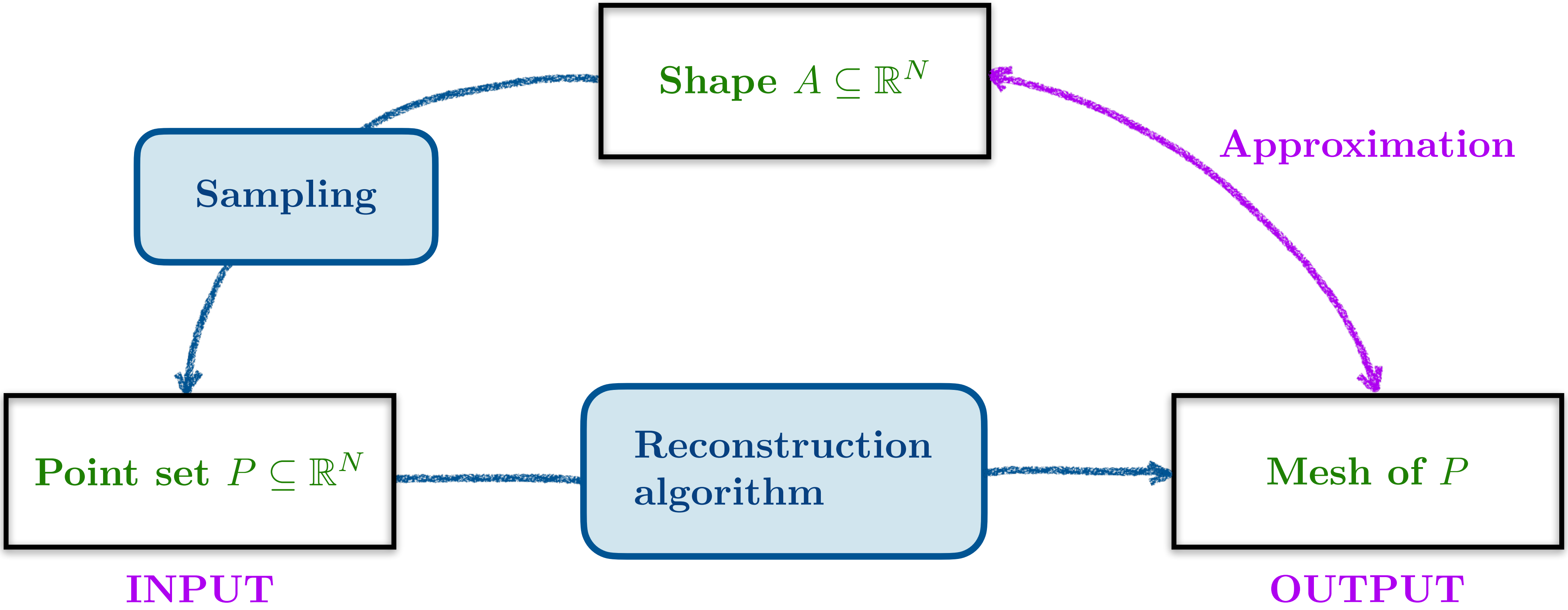
Meshing manifolds by weighted $\ell 1$-norm minimization



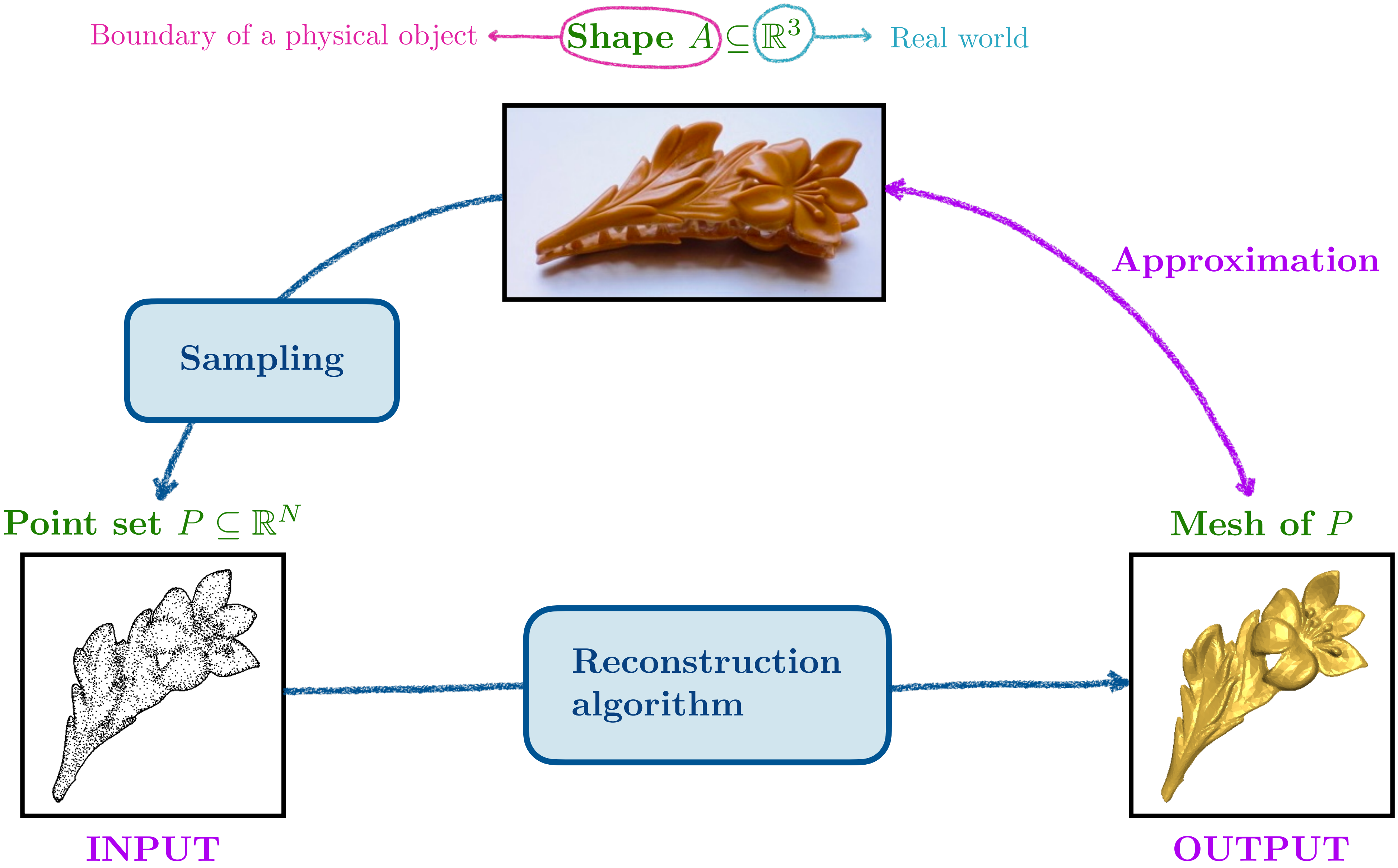
Dominique Attali
GIPSA-lab, CNRS, Grenoble

Joined work with André Lieutier
Dassault systèmes

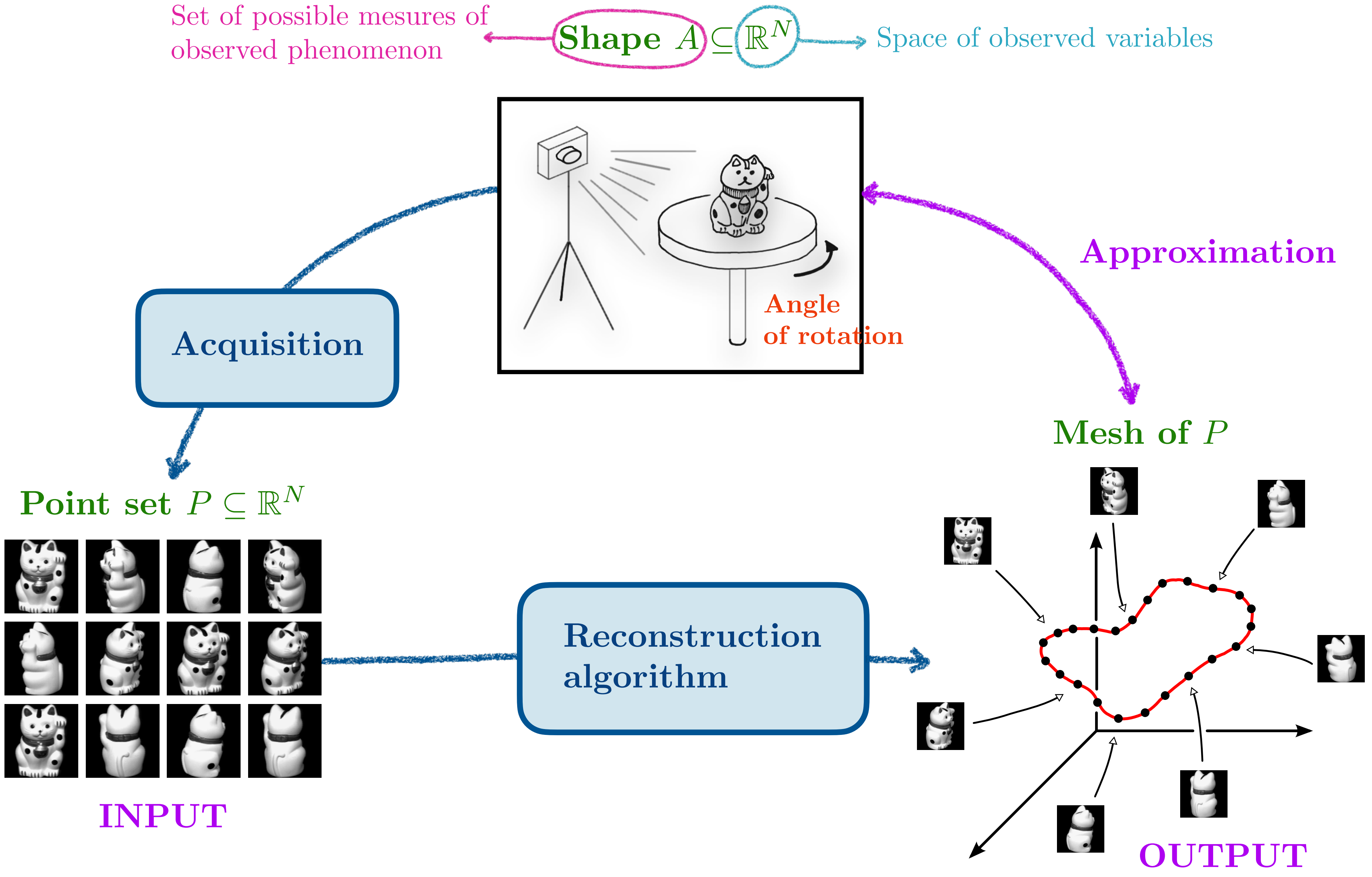
Shape reconstruction problem



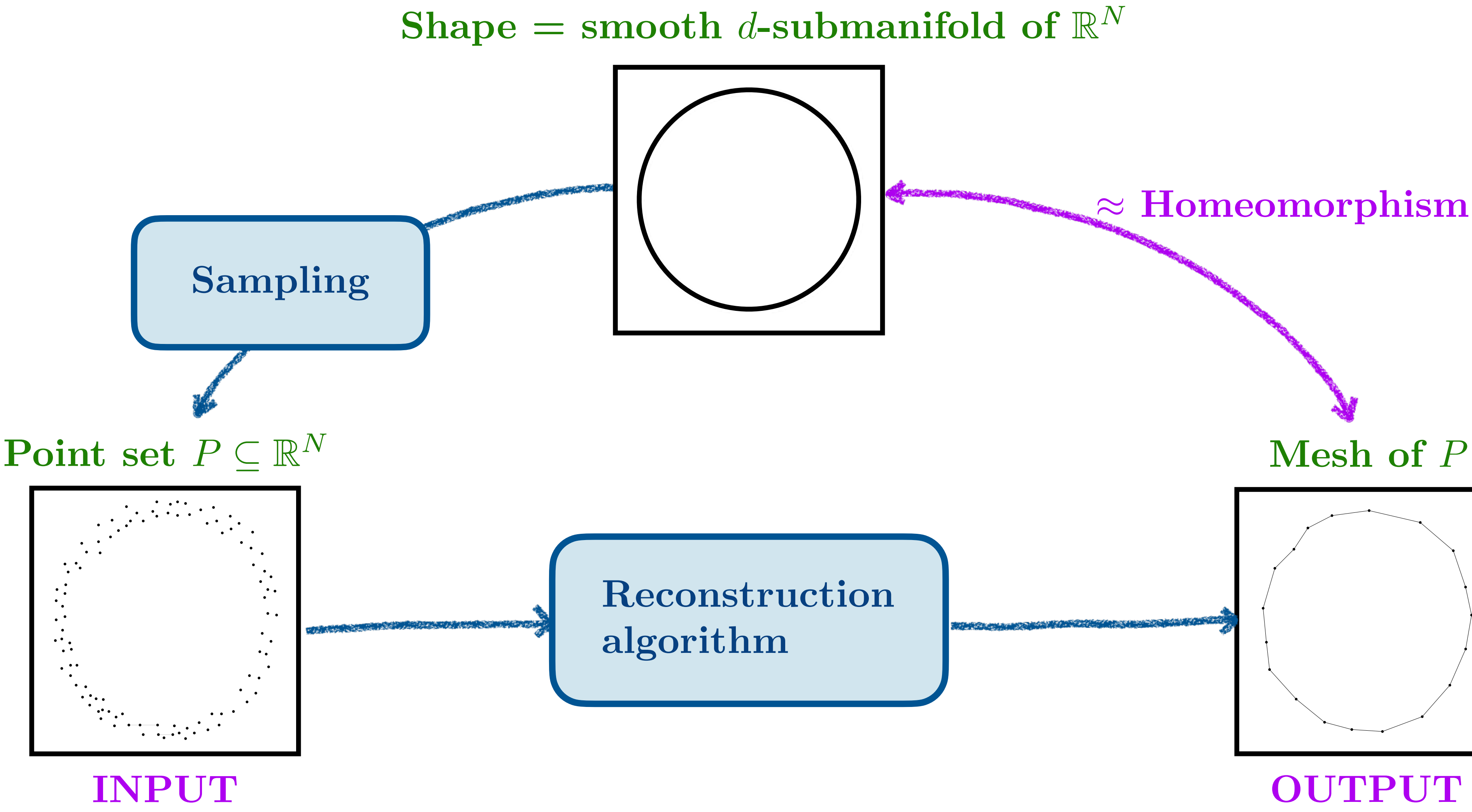
Shape reconstruction for N=3



Shape reconstruction for N large

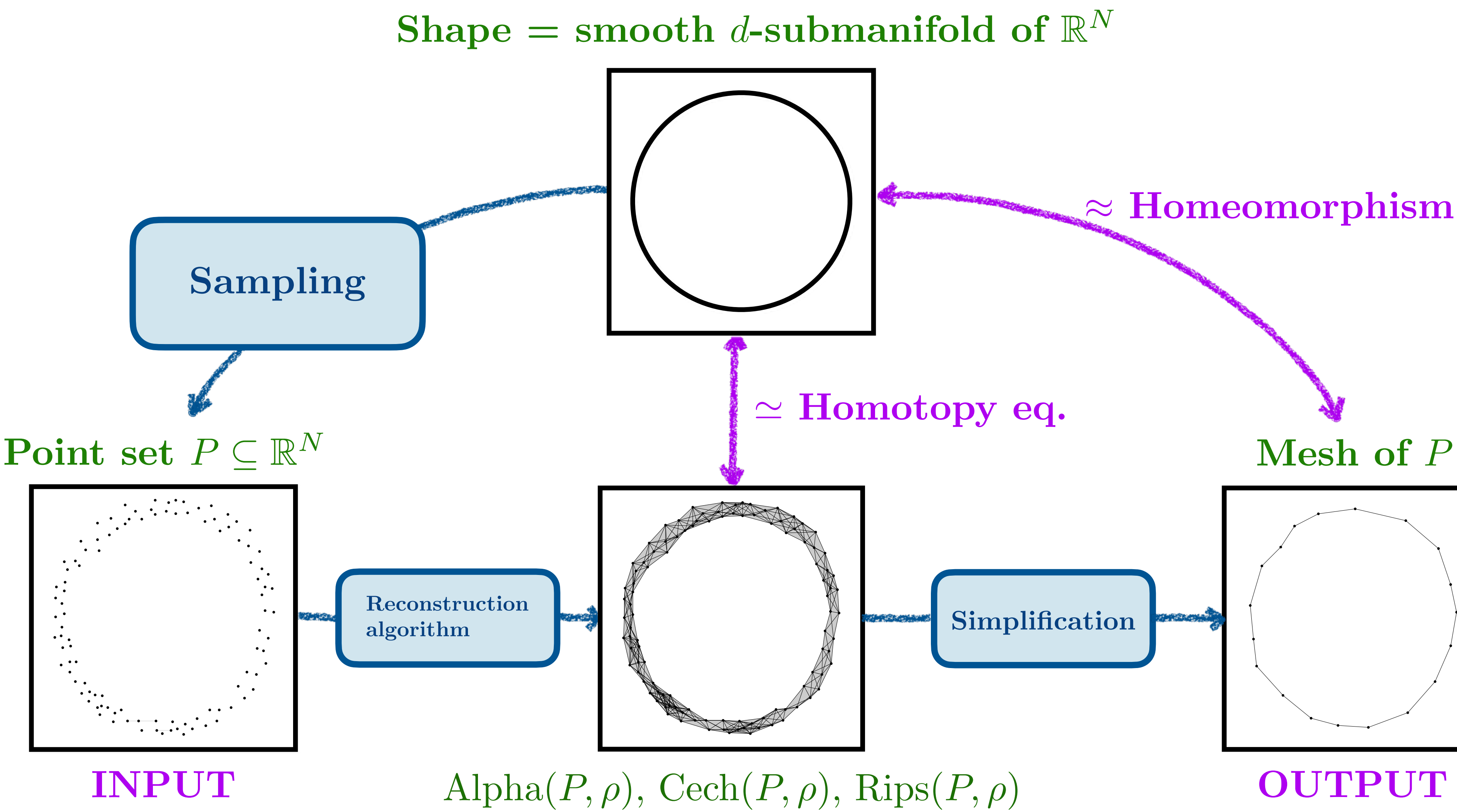


Shape reconstruction problem



Goal: Find conditions under which the output is a triangulation of the shape

Shape reconstruction problem



Goal: Find conditions under which the output is a triangulation of the shape

Manifold reconstruction problem

Different strategies:

1. ~~**Through simplification (collapses, contractions,…)**~~

   [A. & Lieutier 2015][A., Lieutier & Salinas 2012]

2. **Direct approach**

   [Boissonnat, Dyer, Ghosh, Lieutier, Wintraecken 2019][Boissonnat, Dyer, Ghosh 2017]
   [Boissonnat, Ghosh 2010][Boissonnat, Flötotto 2004]

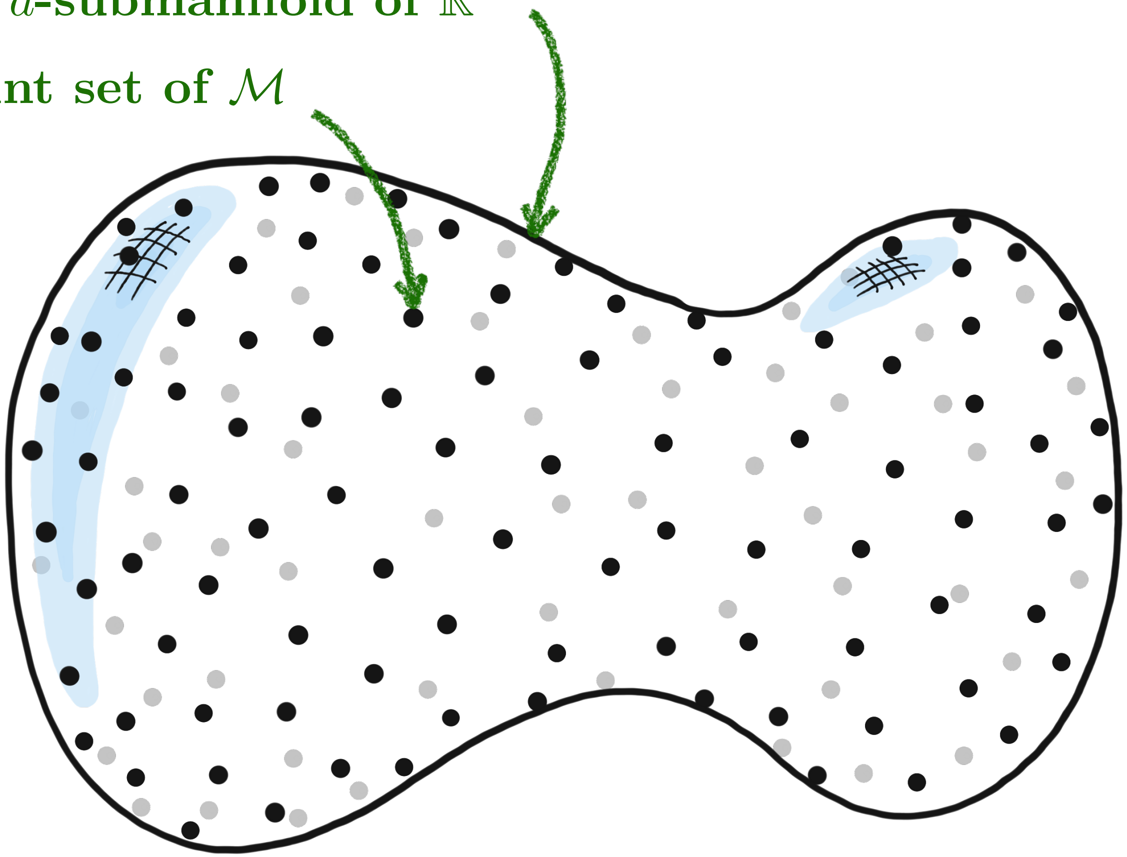
3. **Through energy minimization**

   [Chen, Holst 2011][Alliez, Cohen-Steiner, Yvinec, Desbrun 2005]
   [Rakovic, Grieder & Jones 2004][Musin 2003]

Manifold reconstruction problem

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

P: a finite point set of $\mathcal{M}$

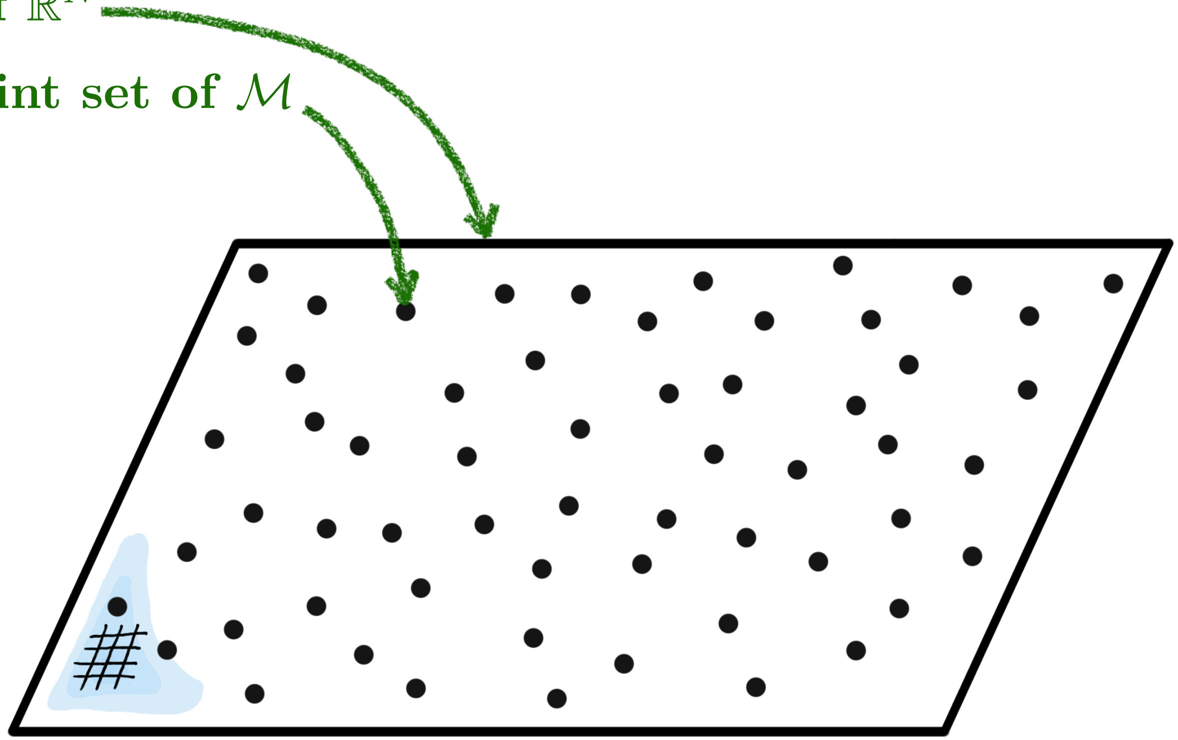


How to triangulate $\mathcal{M}$ given as input P?

Manifold reconstruction problem

$\mathcal{M}$: a d-flat of $\mathbb{R}^N$

P: a finite point set of $\mathcal{M}$



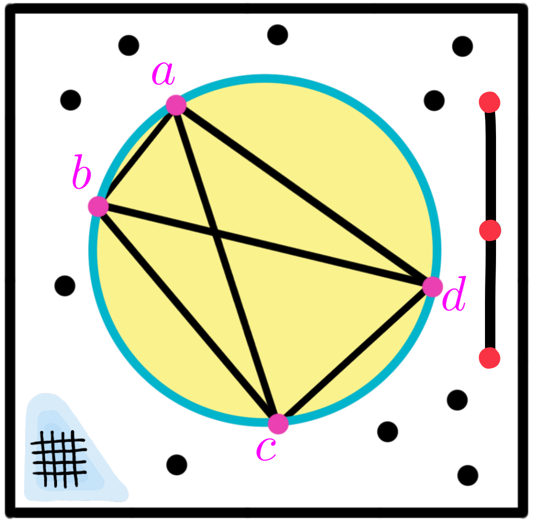
How to triangulate $\mathcal{M}$ given as input P?

Construct the Delaunay complex of P

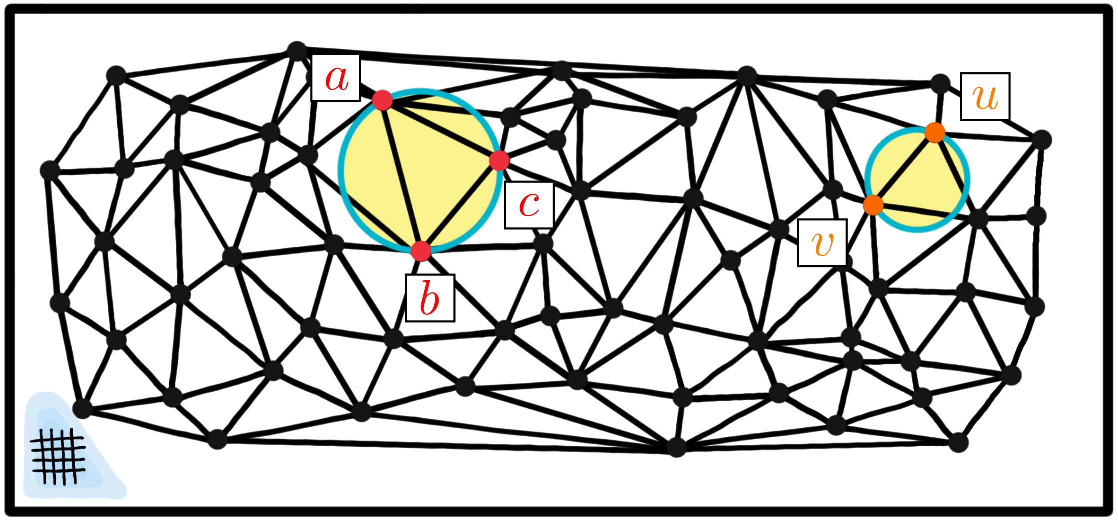
Delaunay complex

P: a finite point set of $\mathbb{R}^d$

$\mathrm{Del}(P) = \{\sigma \mid \emptyset \neq \sigma \subseteq P$ and $\exists$ a d-sphere through σ that does not enclose any point of P $\}$



Non-generic point set



Generic point set P:

No $(d+2)$ points of P lie on a common $(d-1)$-sphere

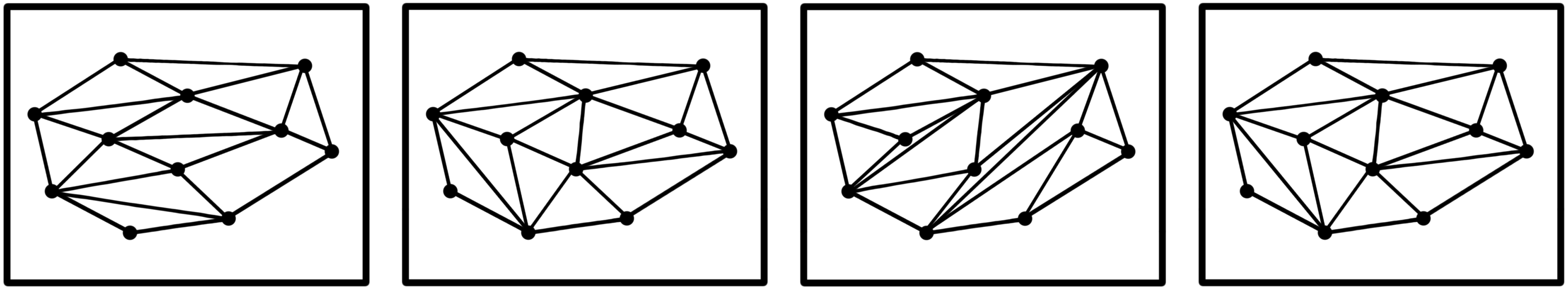
No degenerate simplices of P on $\partial\,\mathrm{Conv}(P)$

$\Downarrow$

$\mathrm{Del}(P)$ triangulation of the convex hull of P

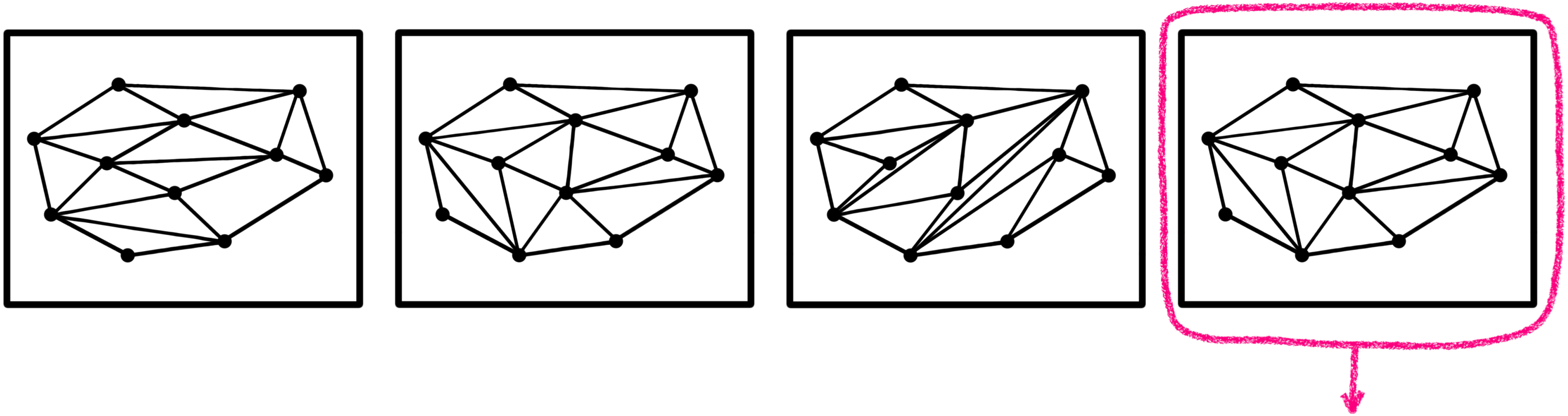
Delaunay complex

Where is the Delaunay complex?



Delaunay complex

Where is the Delaunay complex?



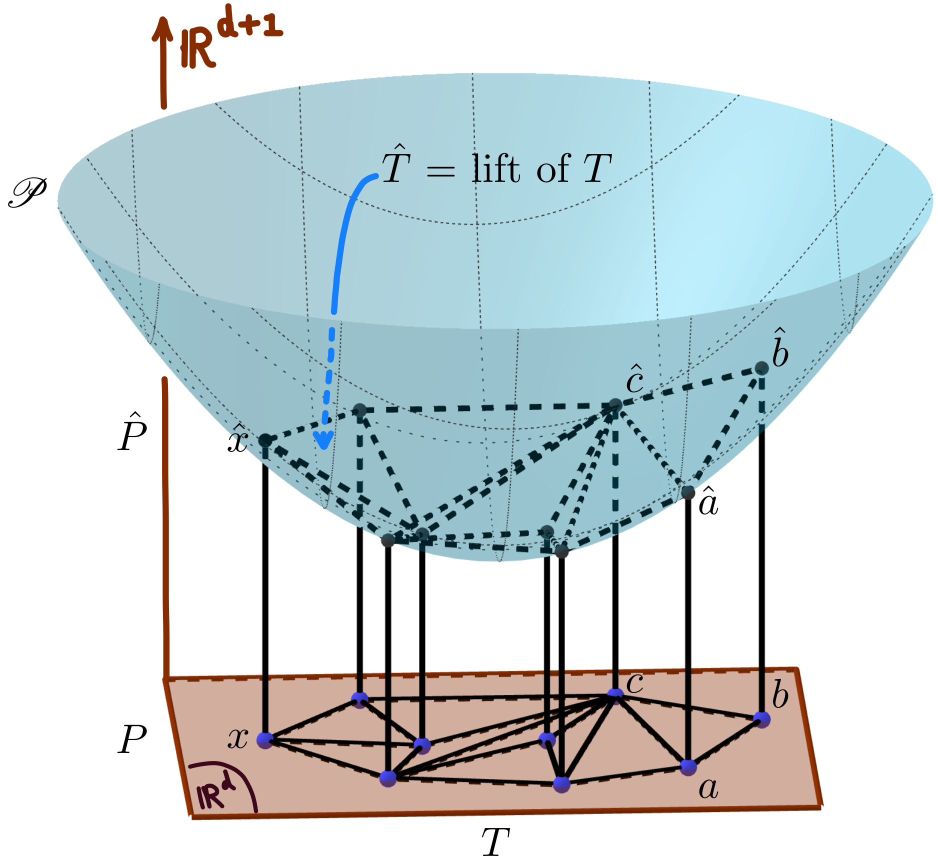
Delaunay complex

(optimizes many functionals)

Delaunay energy

T: triangulation of $\mathrm{Conv}(P)$

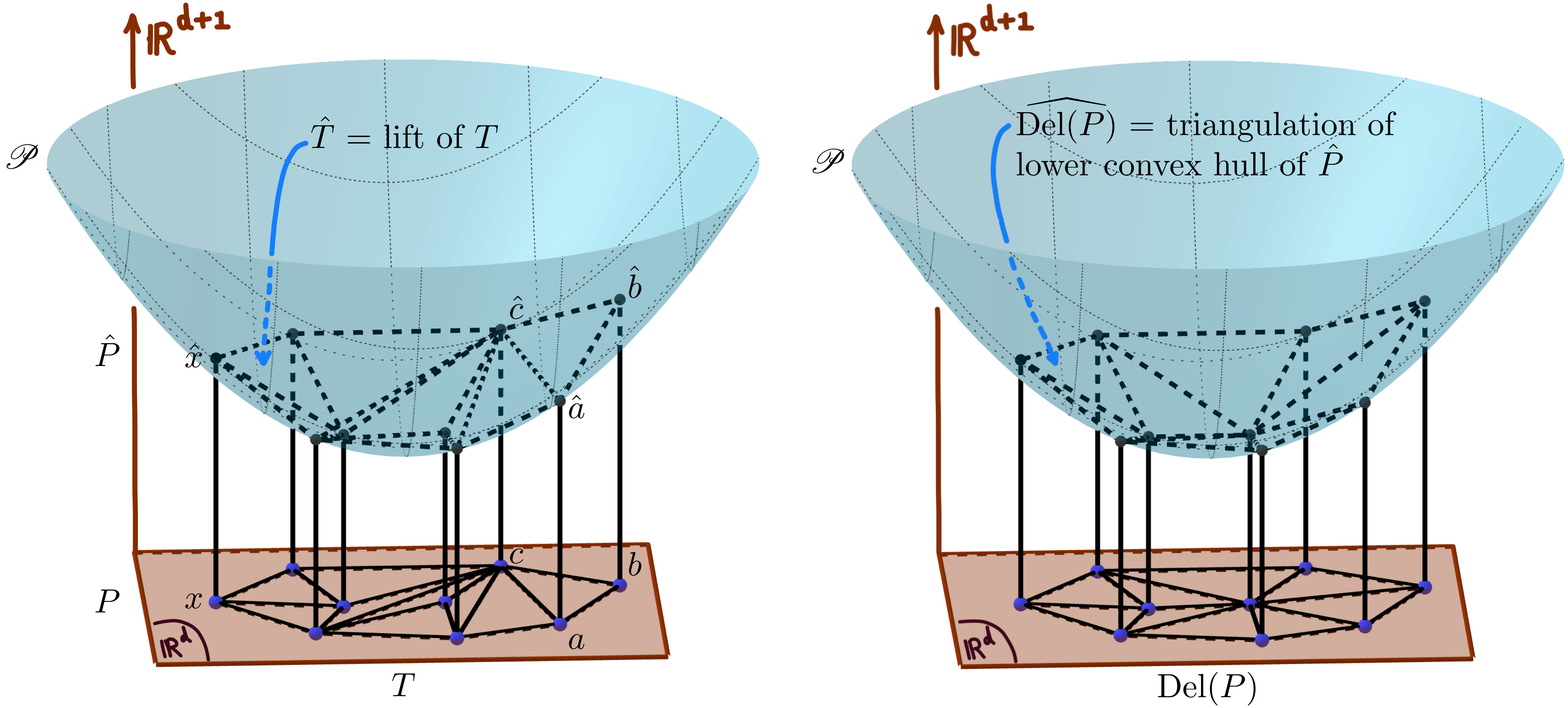
$E_{\mathrm{del}}(T) =$ volume between $\hat{T}$ and paraboloid $\mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}$.



Delaunay energy

T: triangulation of $\mathrm{Conv}(P)$

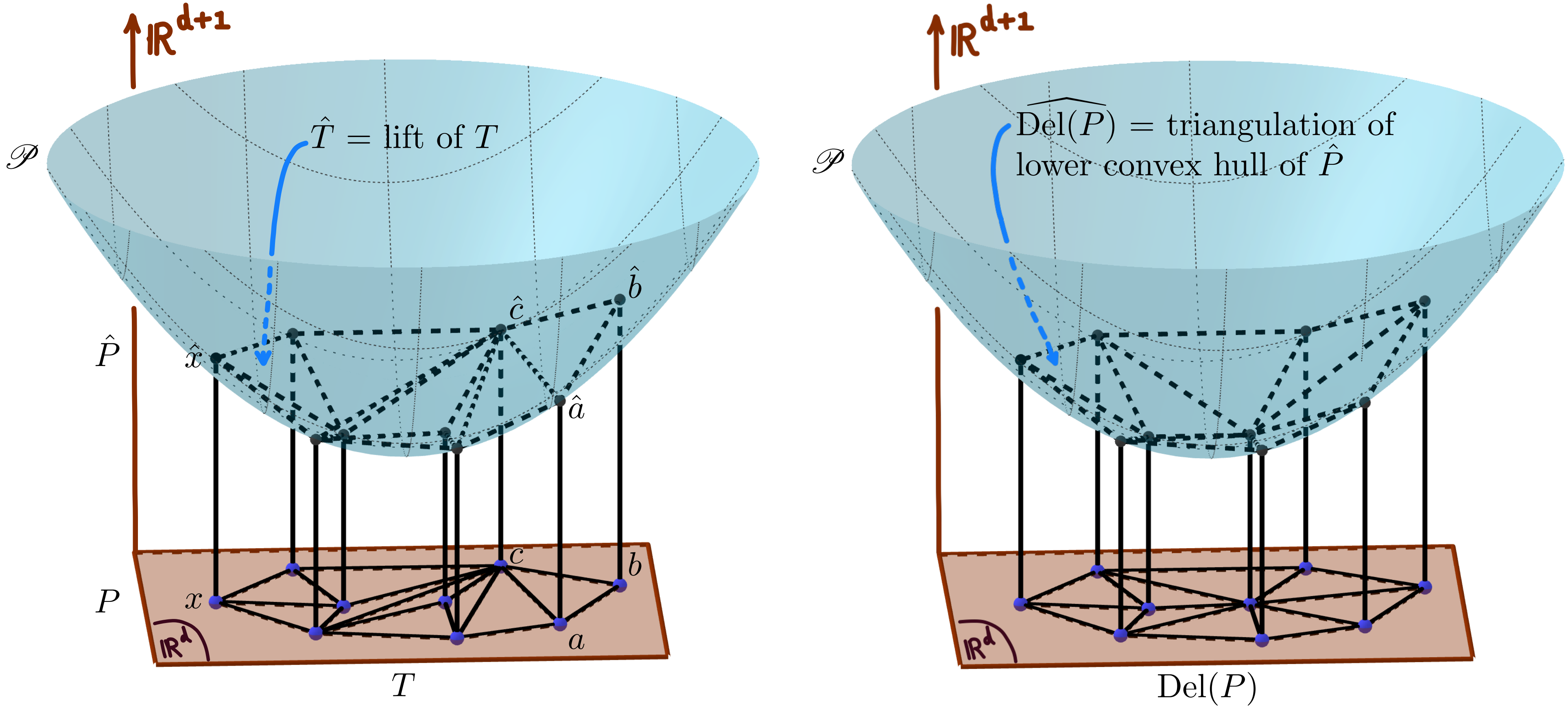
$E_{\mathrm{del}}(T)$ = volume between $\hat{T}$ and paraboloid $\mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}$.



Delaunay energy

T: triangulation of $\mathrm{Conv}(P)$

$E_{\mathrm{del}}(T)$ = volume between $\hat{T}$ and paraboloid $\mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}$.

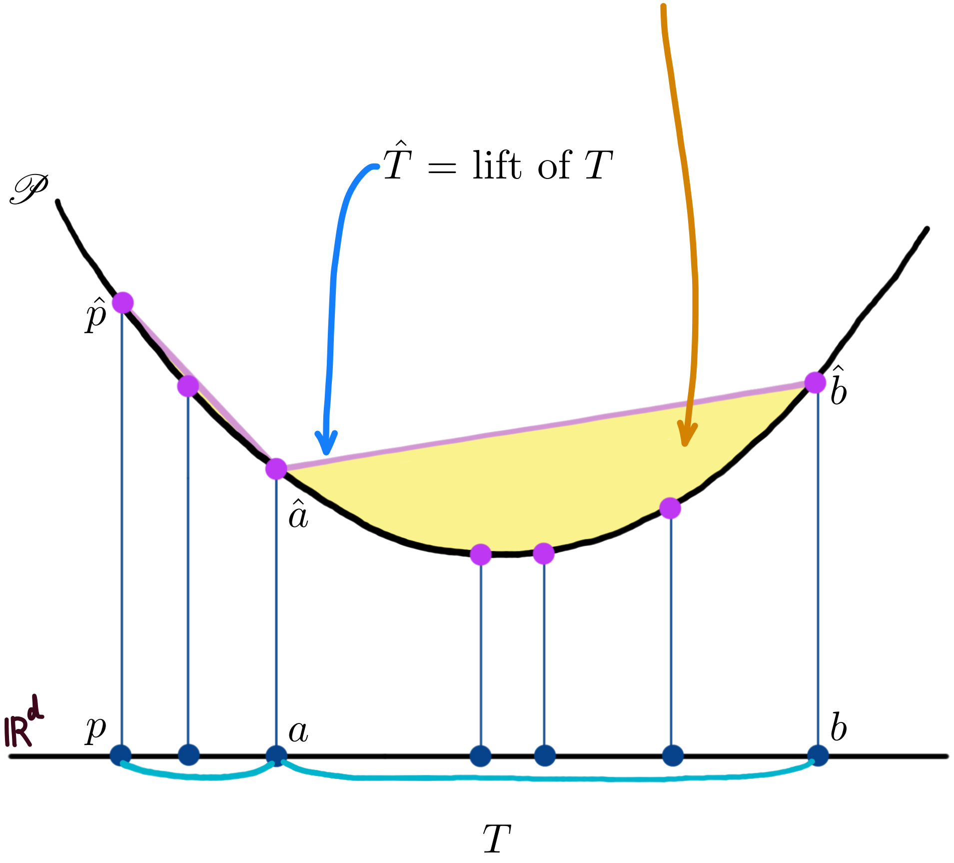


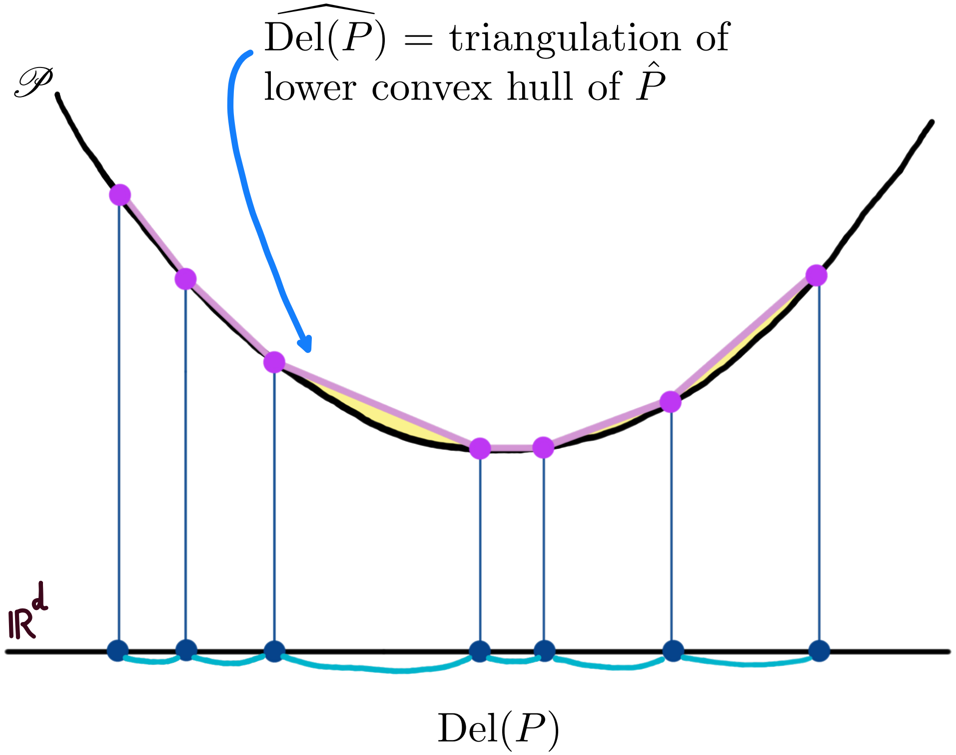
P **generic** $\Longrightarrow$ $\mathrm{Del}(P)$ = the triangulation of $\mathrm{Conv}(P)$ with smallest Delaunay energy

Delaunay energy

T: triangulation of Conv(P)

$E_{\text{del}}(T)$ = volume between $\hat{T}$ and paraboloid $\mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}$.



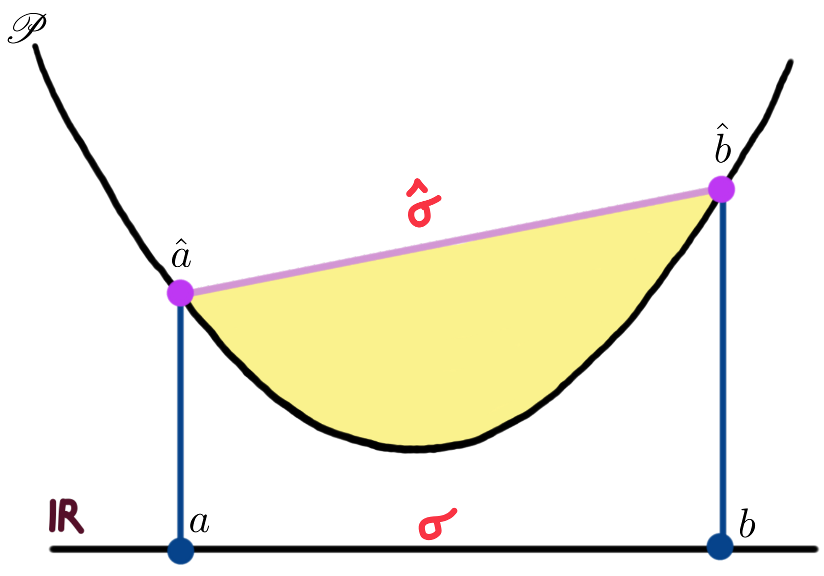


P generic $\Longrightarrow$ Del(P) = the triangulation of Conv(P) with smallest Delaunay energy

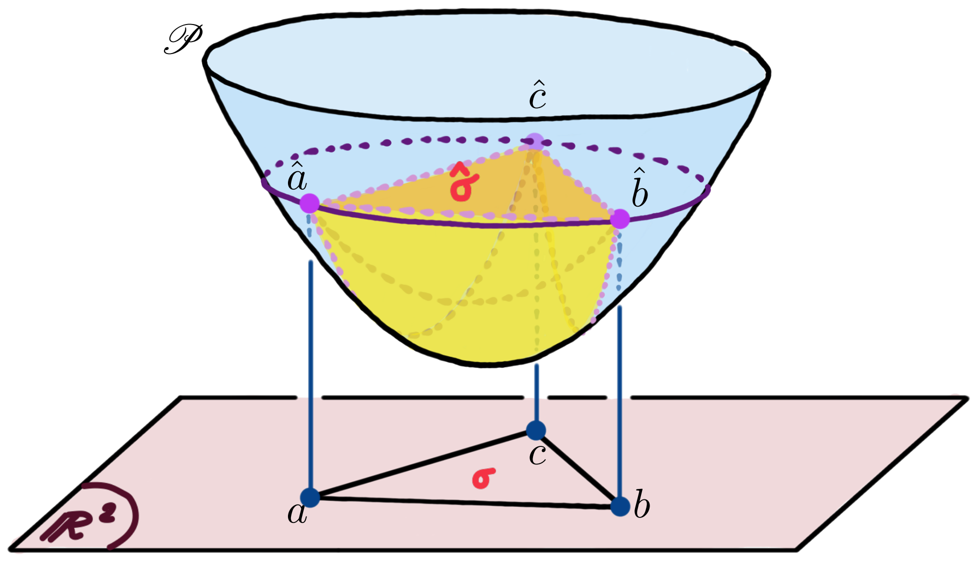
Delaunay energy

T: triangulation of $\mathrm{Conv}(P)$

$E_{\mathrm{del}}(T)$ = volume between $\hat{T}$ and paraboloid $\mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}$.



$\omega_{\mathrm{del}}(ab) = \frac{1}{6}\|a-b\|^3$



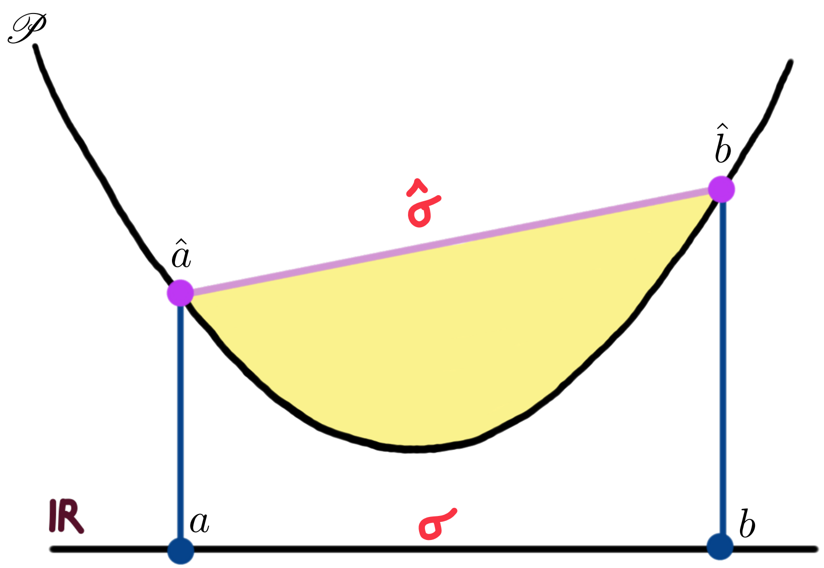
$\omega_{\mathrm{del}}(abc) = \frac{1}{12}\,\mathrm{area}(abc)\left[\|a-b\|^2 + \|b-c\|^2 + \|c-a\|^2\right]$

Delaunay energy

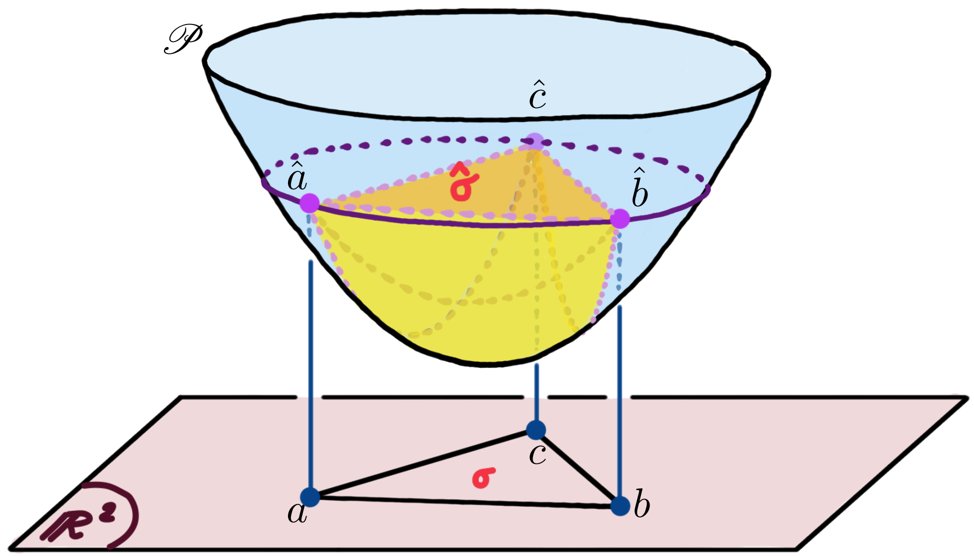
T: triangulation of Conv(P)

$E_{\text{del}}(T) = \text{volume between } \hat{T} \text{ and paraboloid } \mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}.$

$$= \sum_{d\text{-simplex } \sigma \in T} \text{volume between } \hat{\sigma} \text{ and } \mathcal{P}$$



$\omega_{\text{del}}(ab) = \frac{1}{6}\|a-b\|^3$



$\omega_{\text{del}}(abc) = \frac{1}{12}\,\text{area}(abc)\,\left[\|a-b\|^2 + \|b-c\|^2 + \|c-a\|^2\right]$

Delaunay energy

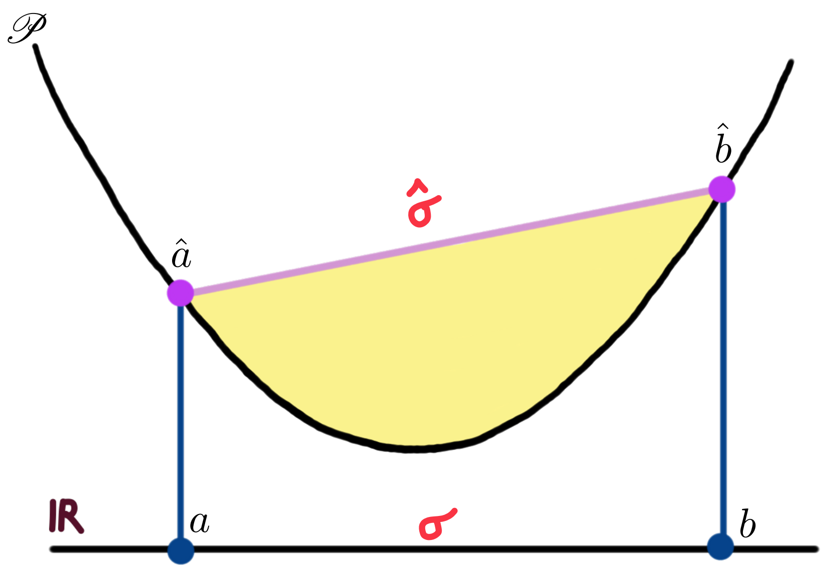
T: triangulation of $\mathrm{Conv}(P)$

$E_{\mathrm{del}}(T)$ = volume between $\hat{T}$ and paraboloid $\mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}$.

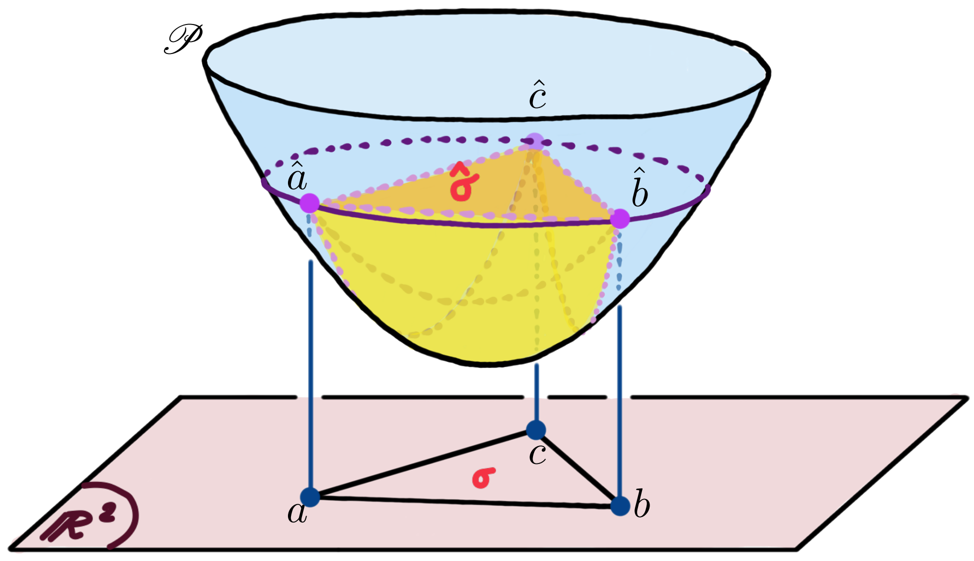
$$= \sum_{d\text{-simplex } \sigma \in T} \text{volume between } \hat{\sigma} \text{ and } \mathcal{P}$$

Delaunay weight of σ = $\frac{1}{(d+1)(d+2)} \mathrm{vol}(\sigma) \sum_{e \text{ edge of } \sigma} \mathrm{length}(e)^2$

intrinsec expression [Chen, Holst 2011]



$\omega_{\mathrm{del}}(ab) = \frac{1}{6}\|a-b\|^3$



$\omega_{\mathrm{del}}(abc) = \frac{1}{12}\mathrm{area}(abc)\left[\|a-b\|^2 + \|b-c\|^2 + \|c-a\|^2\right]$

Delaunay energy

T: triangulation of $\mathrm{Conv}(P)$

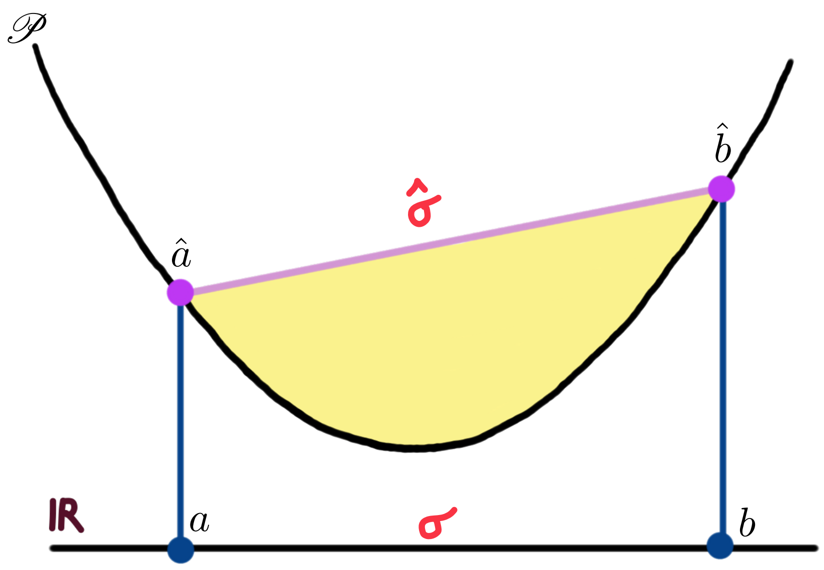
$E_{\mathrm{del}}(T)$ = volume between $\hat{T}$ and paraboloid $\mathscr{P} = \{(x, \|x\|^2) \mid x \in \mathbb{R}^d\}$.

$$= \sum_{d\text{-simplex } \sigma \in T} \text{volume between } \hat{\sigma} \text{ and } \mathcal{P}$$

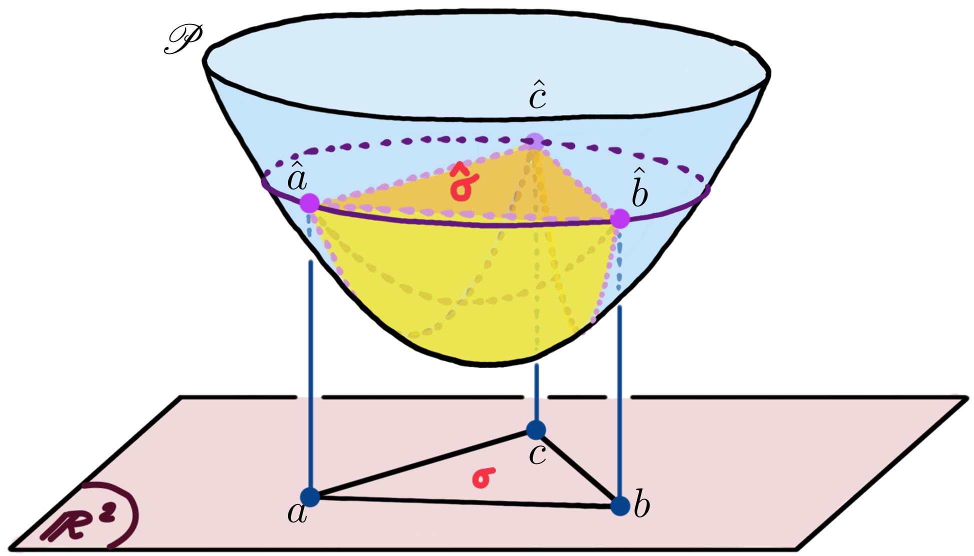
Delaunay weight of σ = $\frac{1}{(d+1)(d+2)} \mathrm{vol}(\sigma) \sum_{e \text{ edge of } \sigma} \mathrm{length}(e)^2$

can be computed for any soup T of d-simplices in $\mathbb{R}^N$

intrinsec expression [Chen, Holst 2011]



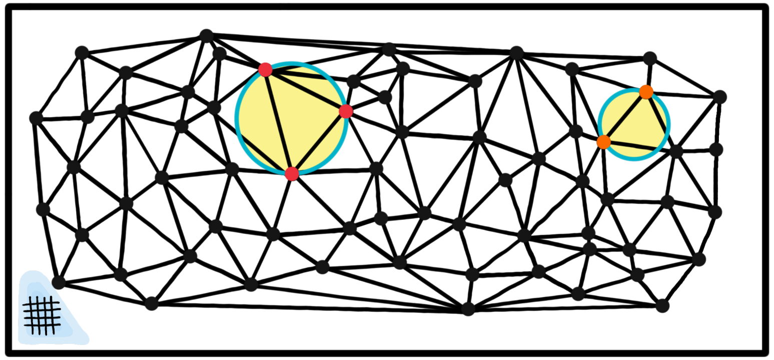
$\omega_{\mathrm{del}}(ab) = \frac{1}{6}\|a-b\|^3$



$\omega_{\mathrm{del}}(abc) = \frac{1}{12}\mathrm{area}(abc)\left[\|a-b\|^2 + \|b-c\|^2 + \|c-a\|^2\right]$

Delaunay complex

$\mathrm{Del}(P)$



Simplicial complex with vertex set P

Geometric characterization of elements

P **generic**

$\Downarrow$

Triangulation of $\mathrm{Conv}(P)$

Variational characterization

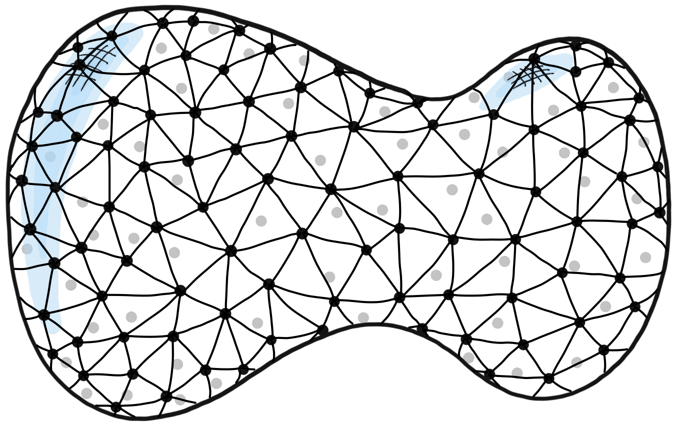
P **generic**

$\Downarrow$

Minimizes Delaunay energy

Delaunay complex generalization

?

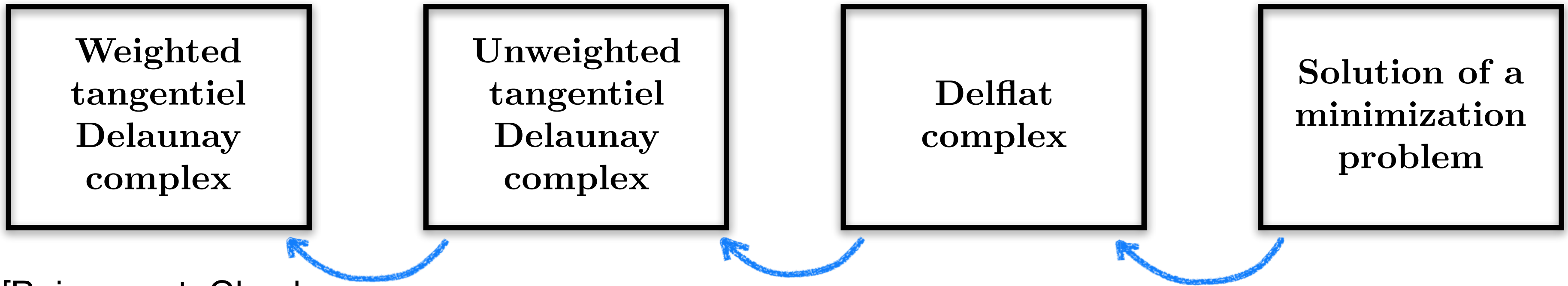


Simplicial complex with vertex set P

Adapt definition	Adapt minimization
Geometric characterization of elements	**Variational characterization**
P ~~generic~~ → P samples $\mathcal{M}$ sufficently "nicely" and sufficiently generic	P ~~generic~~
⇓	⇓
Triangulation of ~~Conv(P)~~ → $\mathcal{M}$	Minimizes Delaunay energy

Road map

1. **Define a Delaunay complex generalization**
2. **Show that indeed triangulates manifold under some conditions**
3. **Define a minimization problem**
4. **Show that indeed solution = Delaunay complex generalization**

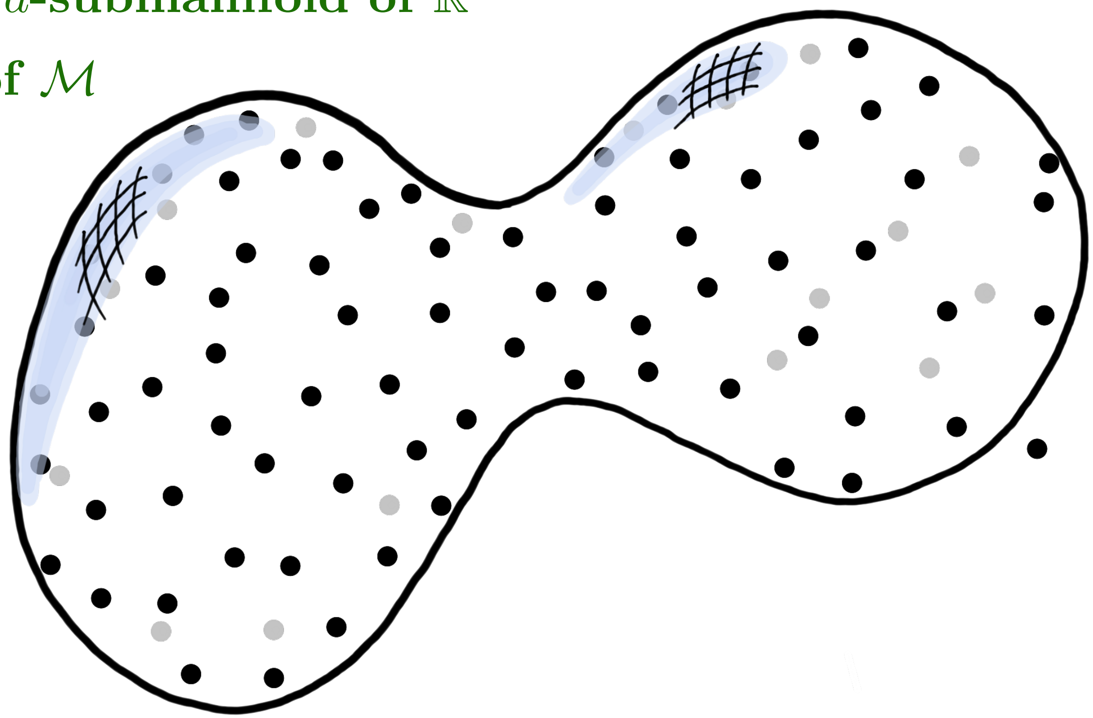


[Boissonnat, Ghosh, Dyer, Wintraecken, Flötotto, Chazal, Yvinec]

Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

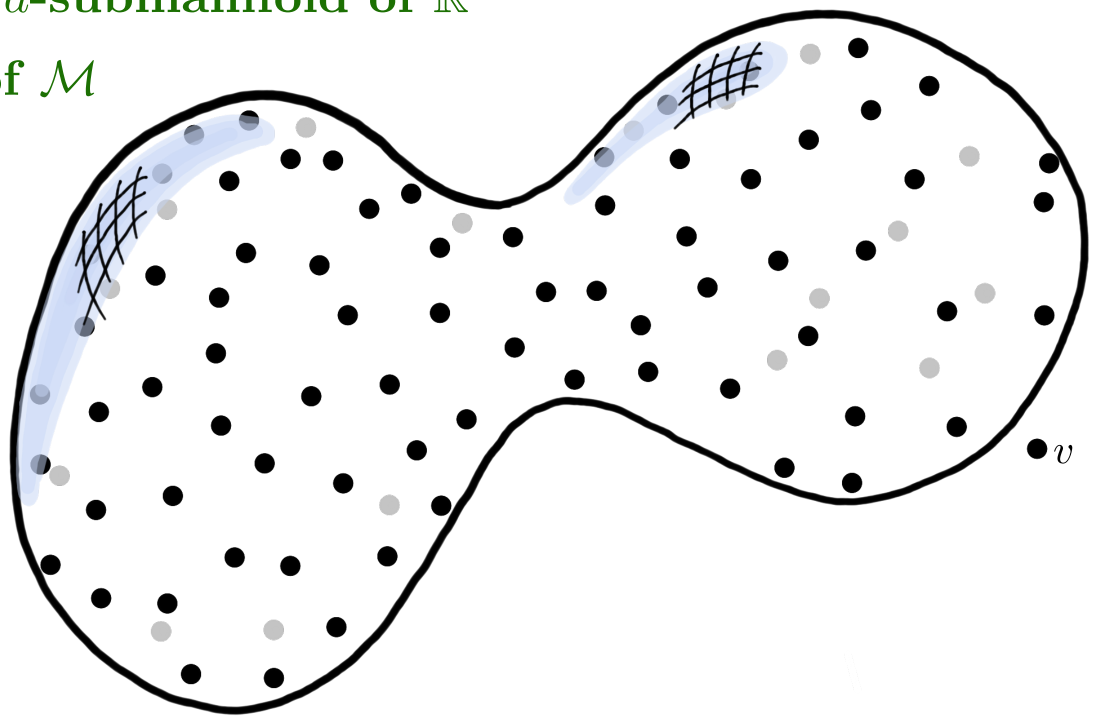
P: a sample of $\mathcal{M}$



Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

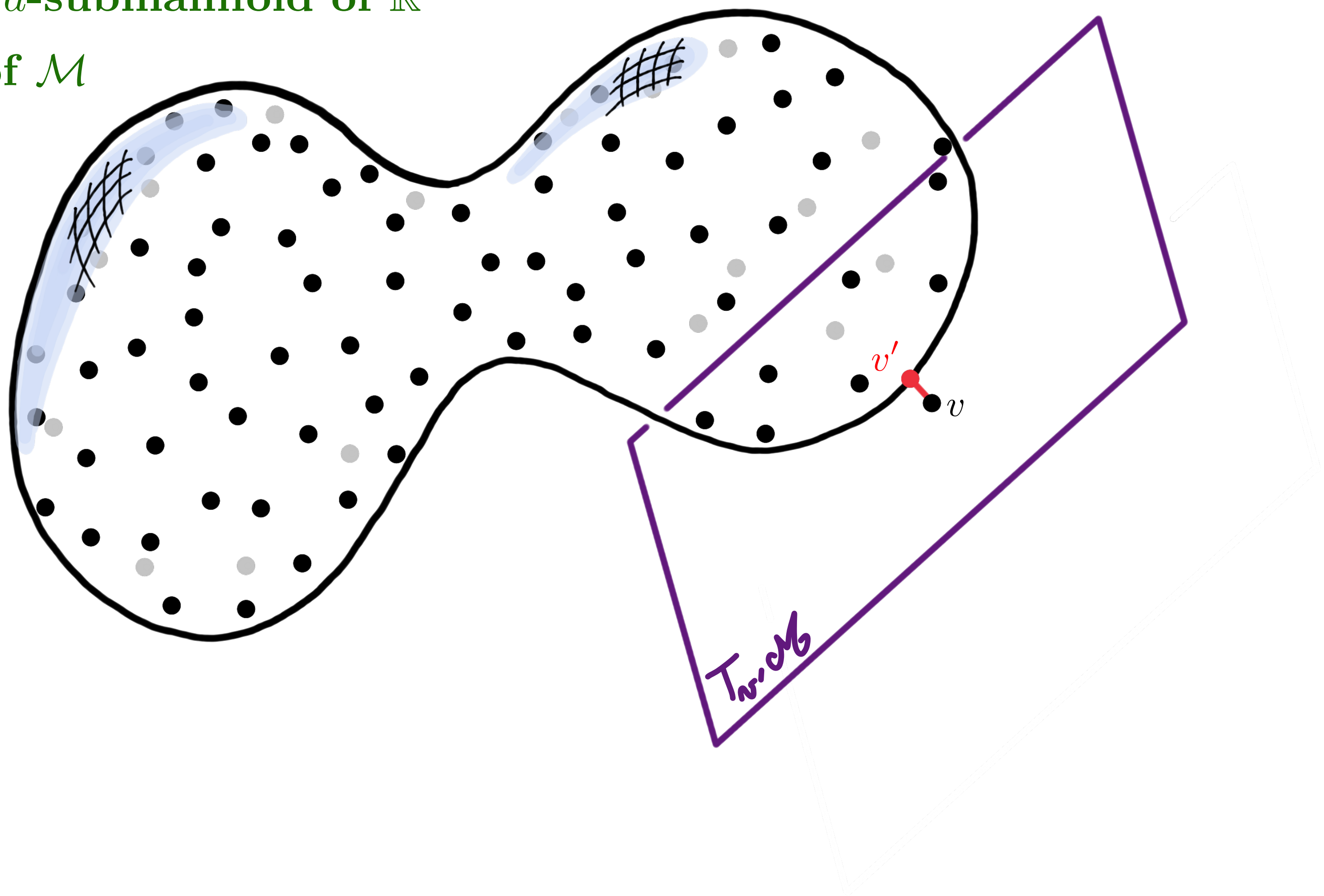
P: a sample of $\mathcal{M}$



Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

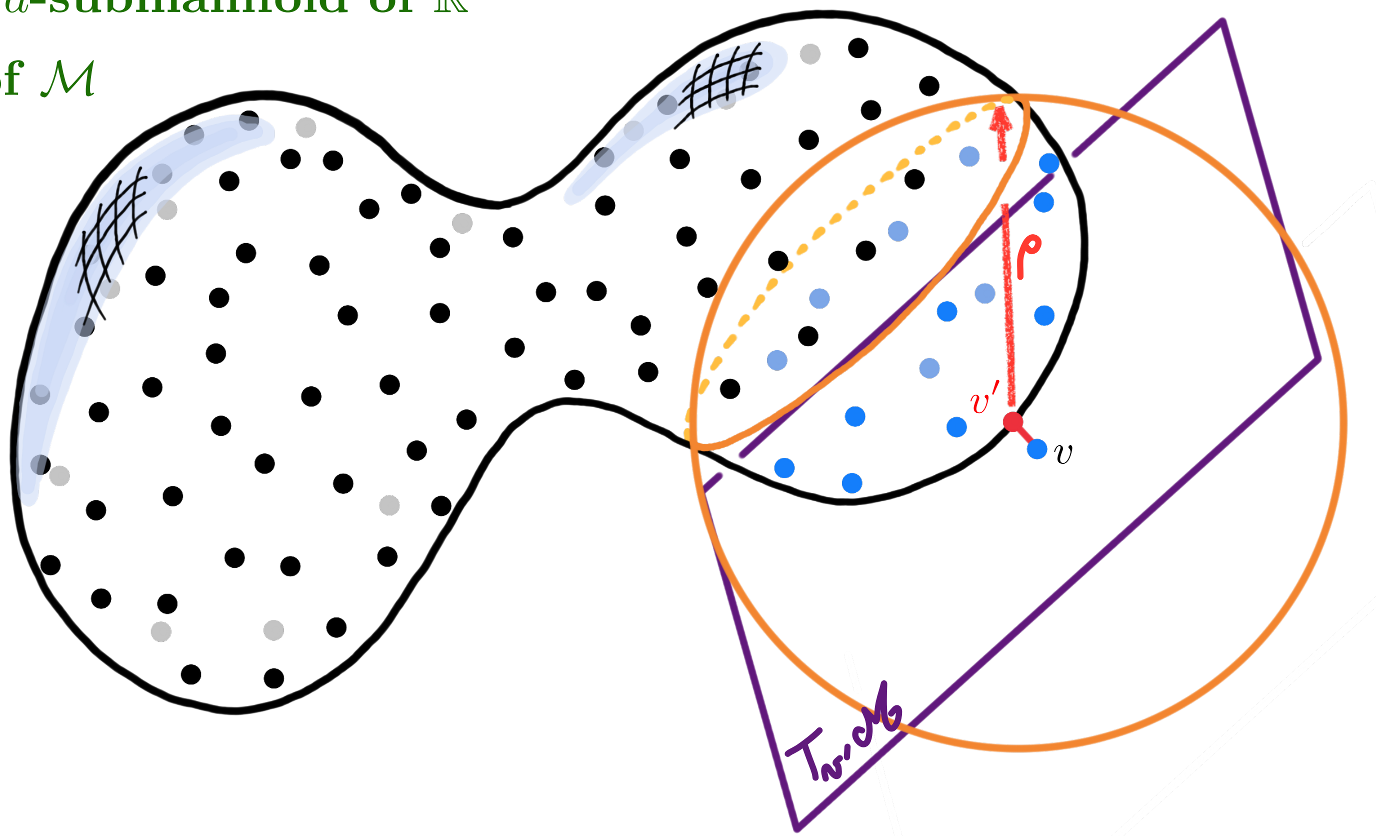
P: a sample of $\mathcal{M}$



Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

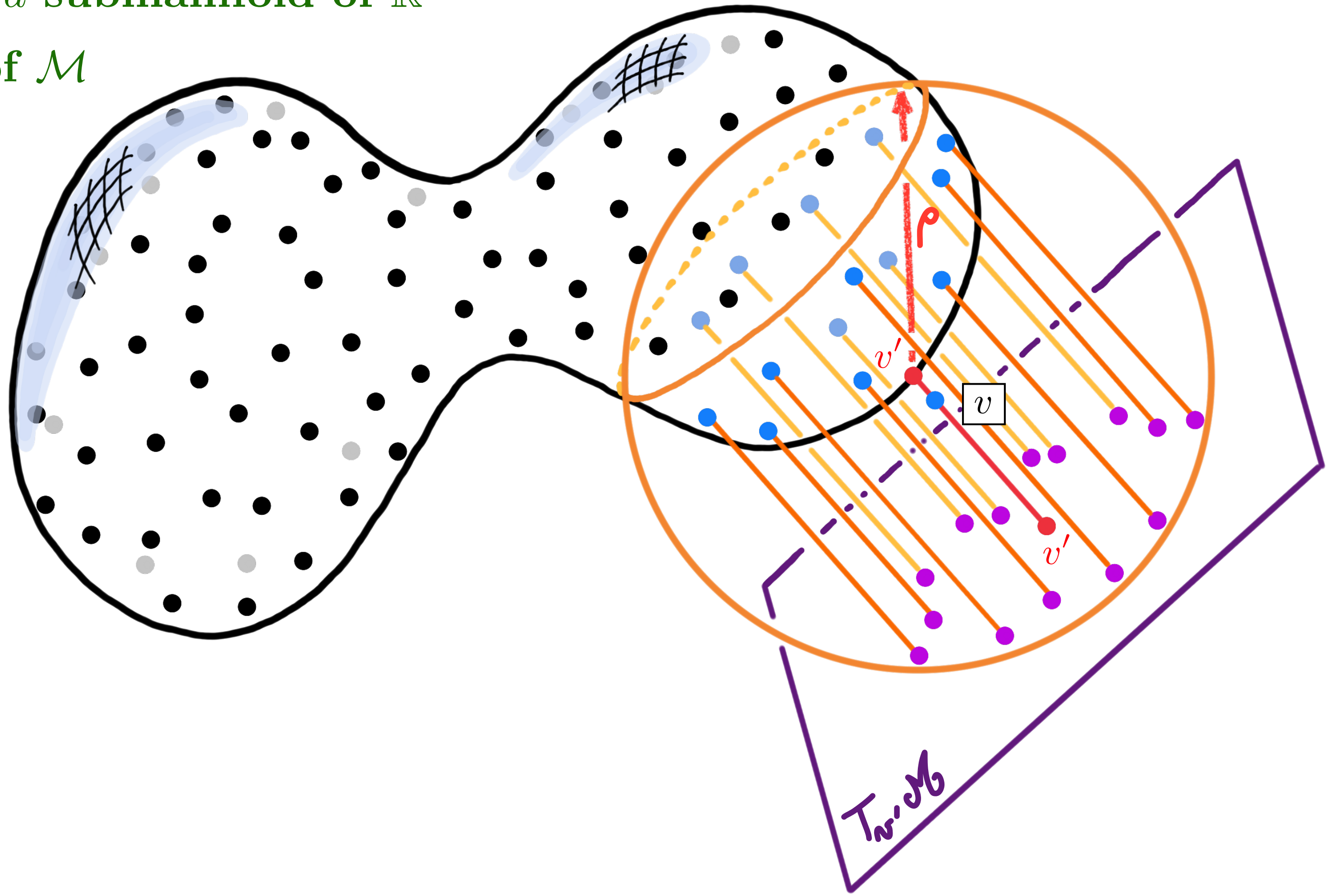
P: a sample of $\mathcal{M}$



Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

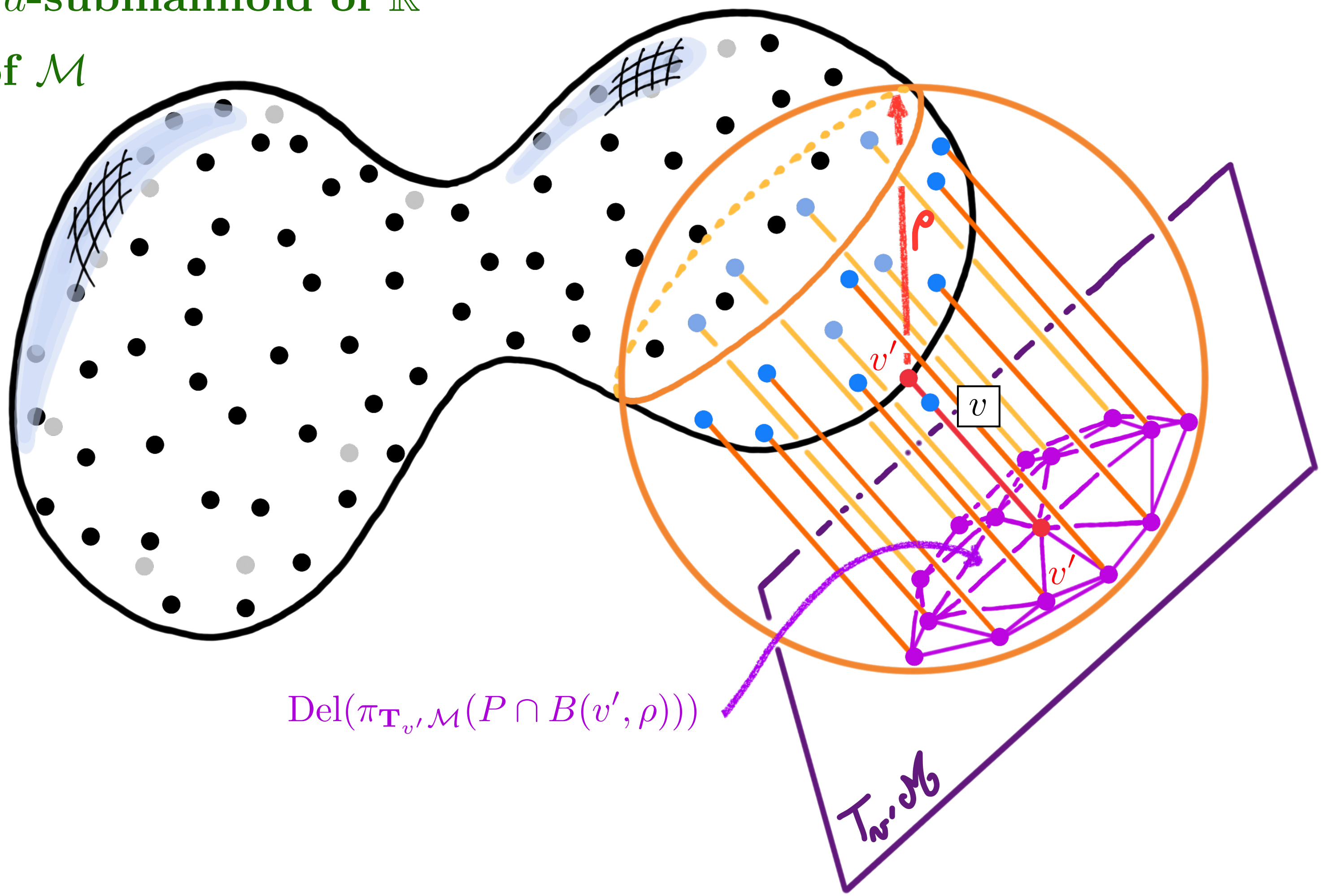
P: a sample of $\mathcal{M}$



Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

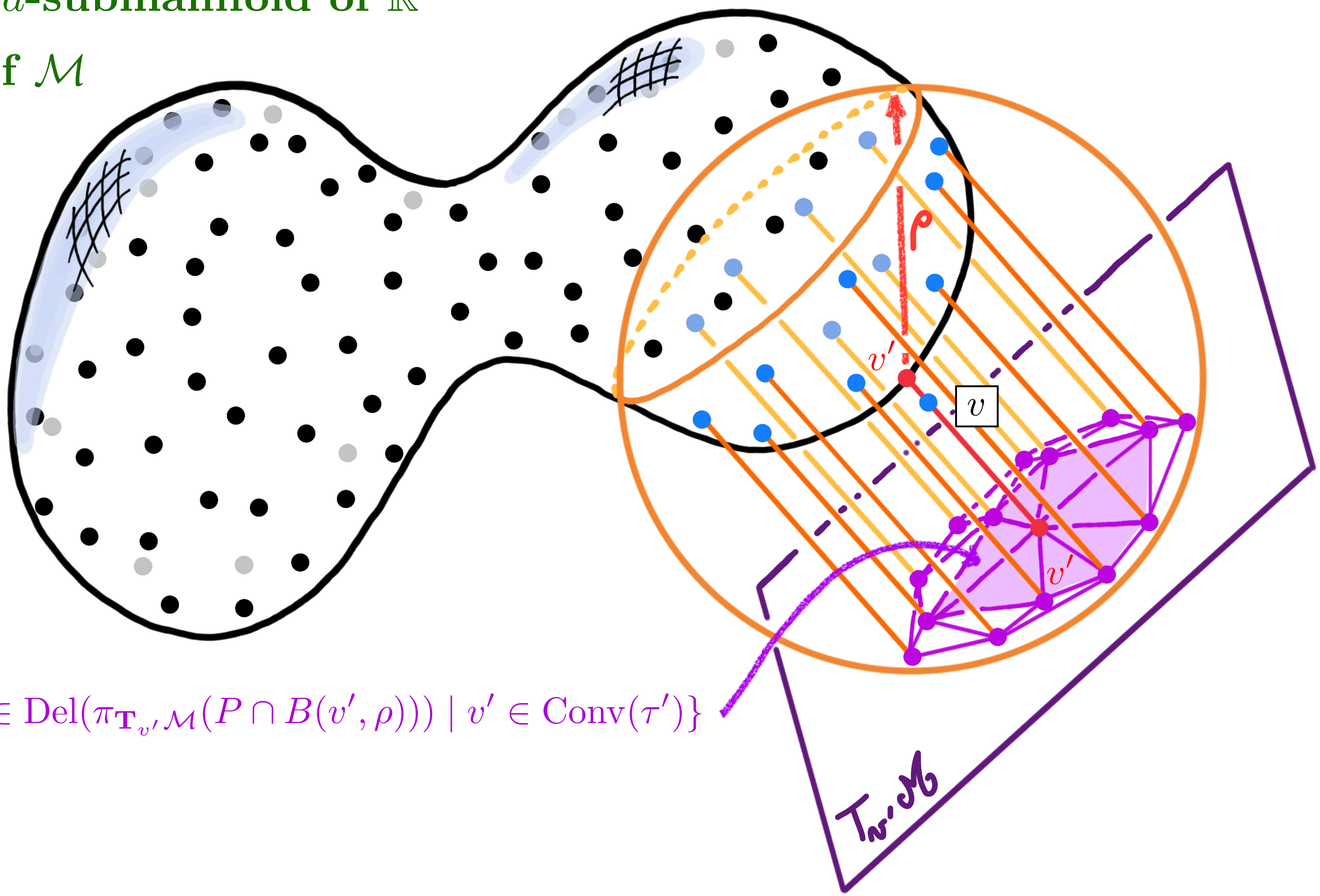
P: a sample of $\mathcal{M}$



Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

P: a sample of $\mathcal{M}$

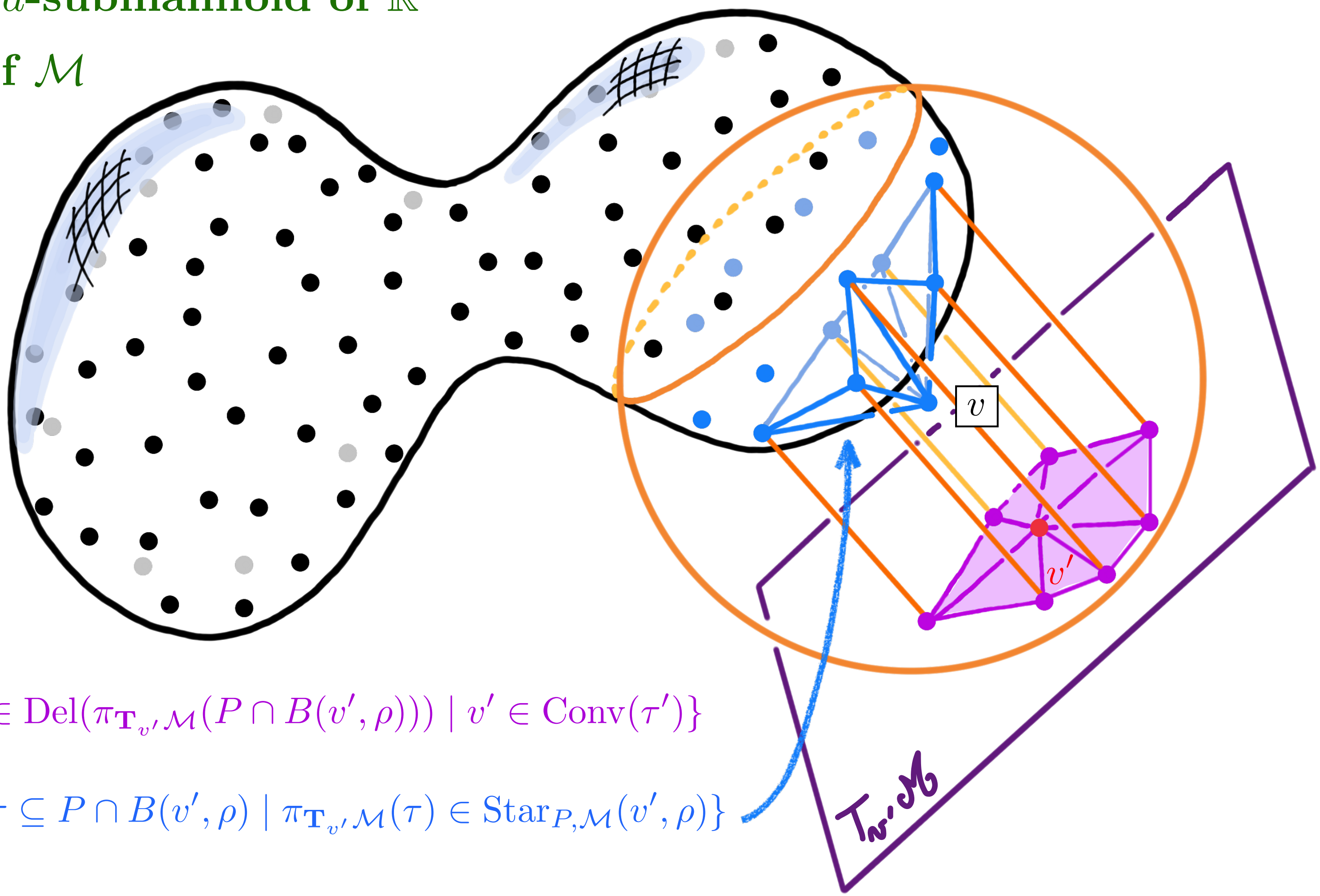


$\text{Star}_{P,\mathcal{M}}(v', \rho) = \{\tau' \in \text{Del}(\pi_{\mathbf{T}_{v'}\mathcal{M}}(P \cap B(v', \rho))) \mid v' \in \text{Conv}(\tau')\}$

Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

P: a sample of $\mathcal{M}$



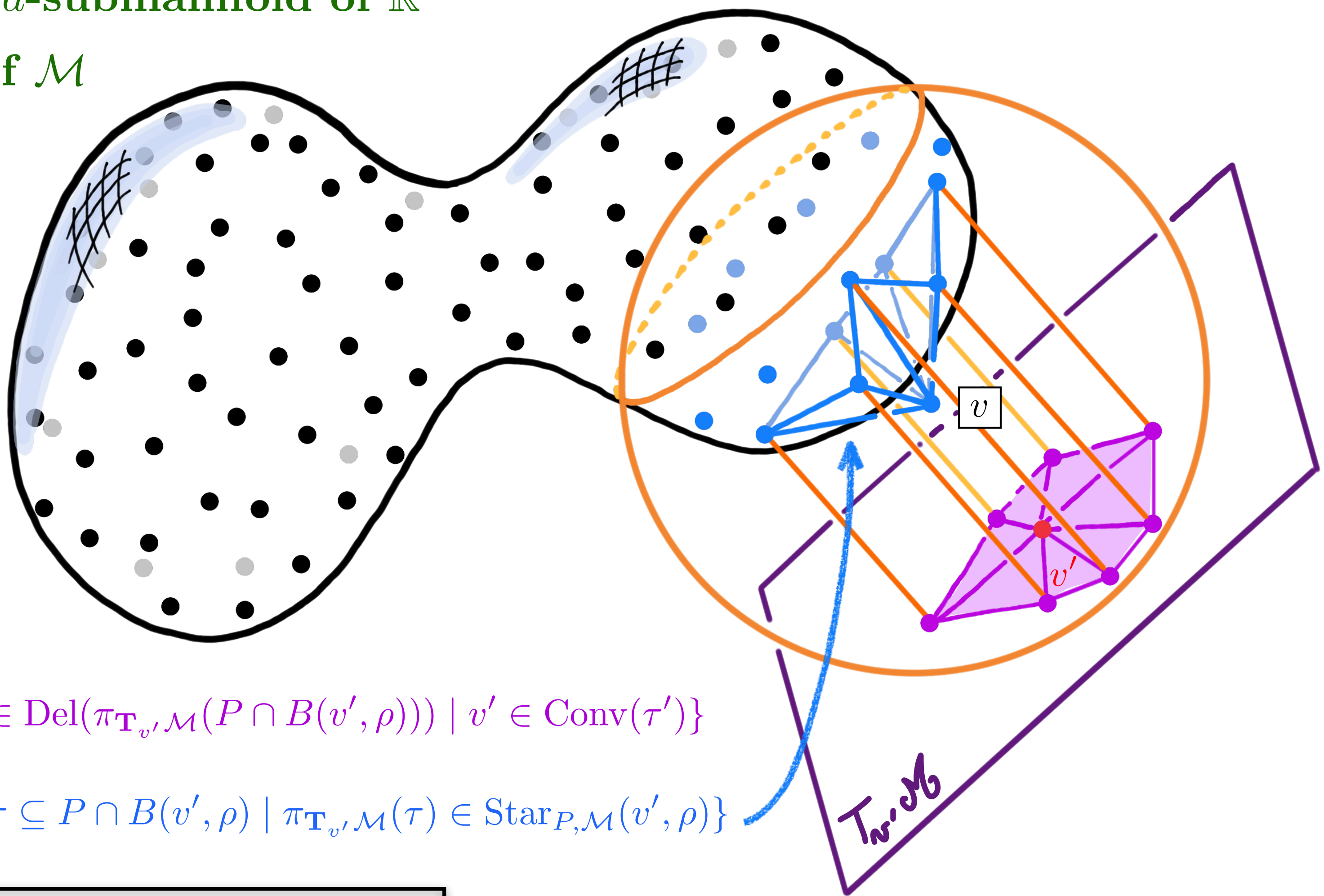
$\mathrm{Star}_{P,\mathcal{M}}(v', \rho) = \{\tau' \in \mathrm{Del}(\pi_{\mathbf{T}_{v'}\mathcal{M}}(P \cap B(v', \rho))) \mid v' \in \mathrm{Conv}(\tau')\}$

$\mathrm{Prestar}_{P,\mathcal{M}}(v, \rho) = \{\tau \subseteq P \cap B(v', \rho) \mid \pi_{\mathbf{T}_{v'}\mathcal{M}}(\tau) \in \mathrm{Star}_{P,\mathcal{M}}(v', \rho)\}$

Unweighted tangentiel Delaunay complex

$\mathcal{M}$: **a smooth d-submanifold of $\mathbb{R}^N$**

P: **a sample of $\mathcal{M}$**



$\mathrm{Star}_{P,\mathcal{M}}(v', \rho) = \{\tau' \in \mathrm{Del}(\pi_{\mathbf{T}_{v'}\mathcal{M}}(P \cap B(v', \rho))) \mid v' \in \mathrm{Conv}(\tau')\}$

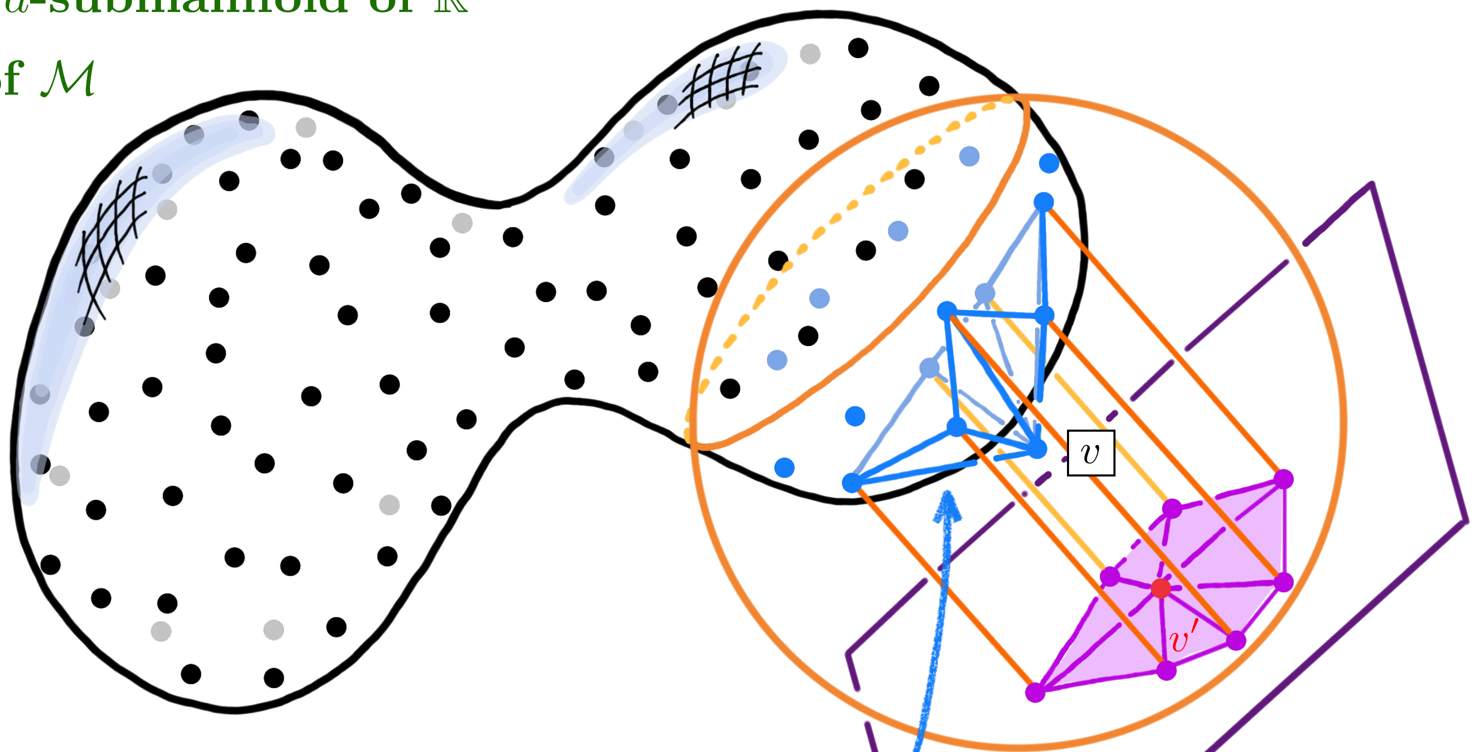
$\mathrm{Prestar}_{P,\mathcal{M}}(v, \rho) = \{\tau \subseteq P \cap B(v', \rho) \mid \pi_{\mathbf{T}_{v'}\mathcal{M}}(\tau) \in \mathrm{Star}_{P,\mathcal{M}}(v', \rho)\}$

$$\mathrm{TangDel}_{\mathcal{M}}(P, \rho) = \bigcup_{v \in P} \mathrm{Prestar}_{P,\mathcal{M}}(v, \rho)$$

Unweighted tangentiel Delaunay complex

$\mathcal{M}$: **a smooth d-submanifold of $\mathbb{R}^N$**

P: **a sample of $\mathcal{M}$**



$\mathrm{Star}_{P,\mathcal{M}}(v',\rho) = \{\tau' \in \mathrm{Del}(\pi_{\mathbf{T}_{v'}\mathcal{M}}(P \cap B(v',\rho))) \mid v' \in \mathrm{Conv}(\tau')\}$

$\mathrm{Prestar}_{P,\mathcal{M}}(v,\rho) = \{\tau \subseteq P \cap B(v',\rho) \mid \pi_{\mathbf{T}_{v'}\mathcal{M}}(\tau) \in \mathrm{Star}_{P,\mathcal{M}}(v',\rho)\}$

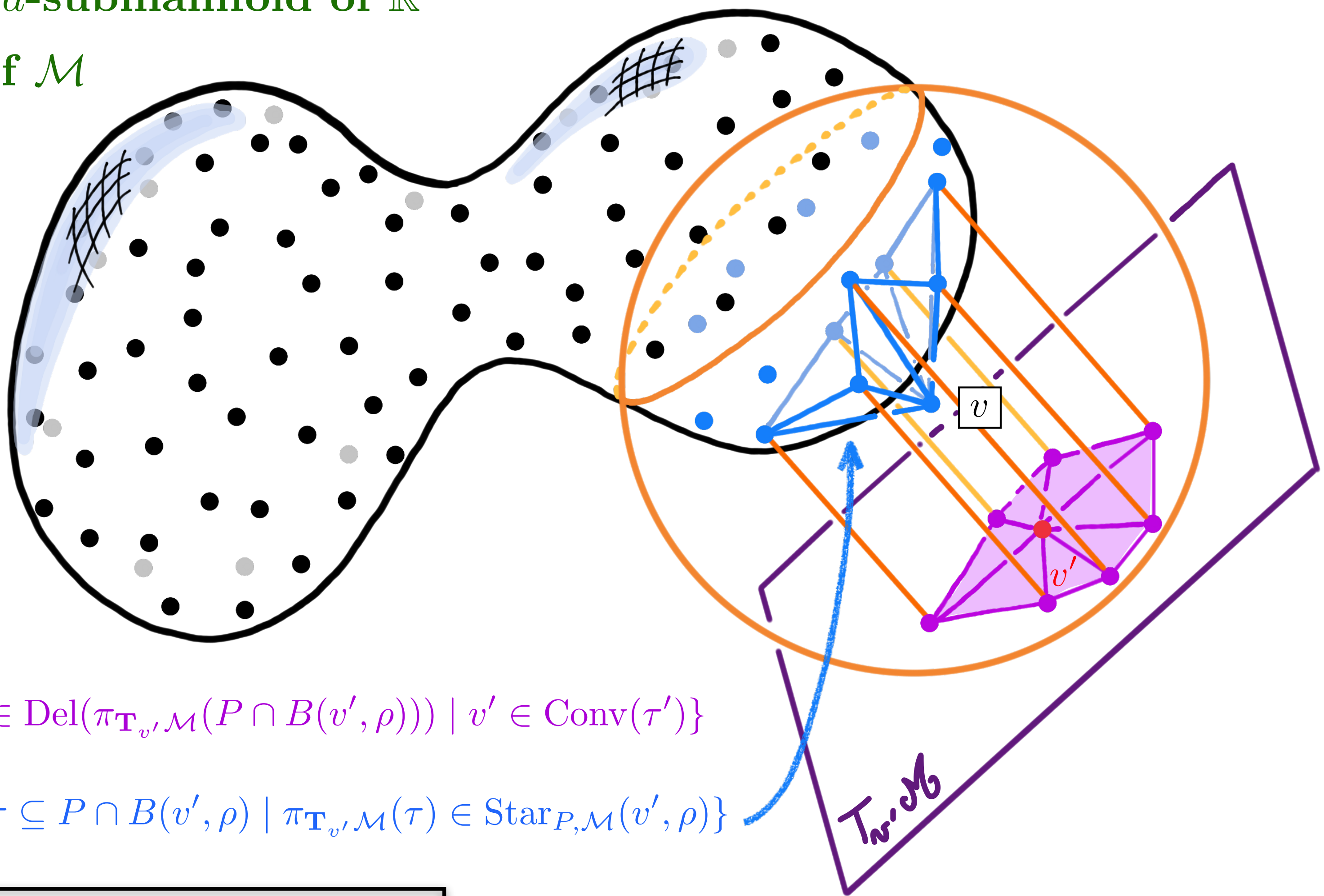
$$\mathrm{TangDel}_{\mathcal{M}}(P,\rho) = \bigcup_{v \in P} \mathrm{Prestar}_{P,\mathcal{M}}(v,\rho)$$

Construction involves computing $|P|$ small d-dimensional Delaunay complexes

Unweighted tangentiel Delaunay complex

$\mathcal{M}$: a smooth d-submanifold of $\mathbb{R}^N$

P: a sample of $\mathcal{M}$



$\mathrm{Star}_{P,\mathcal{M}}(v',\rho) = \{\tau' \in \mathrm{Del}(\pi_{\mathbf{T}_{v'}\mathcal{M}}(P \cap B(v',\rho))) \mid v' \in \mathrm{Conv}(\tau')\}$

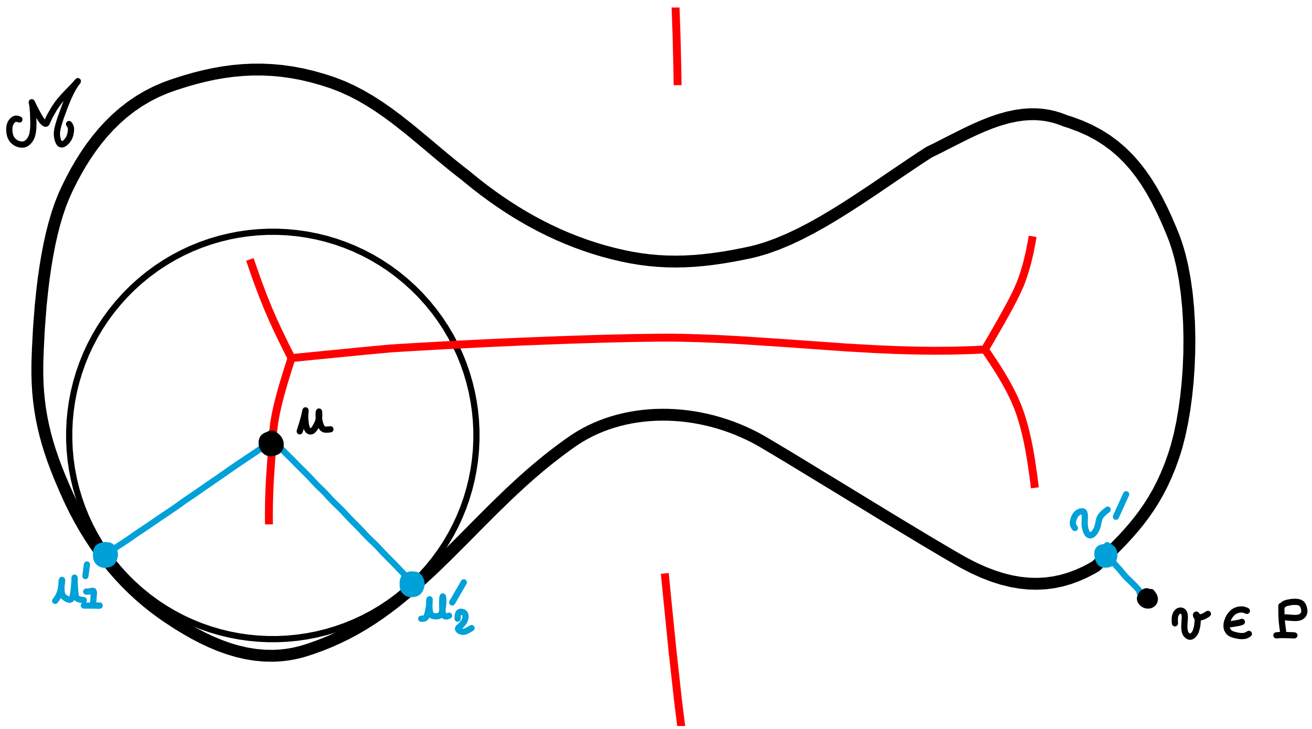
$\mathrm{Prestar}_{P,\mathcal{M}}(v,\rho) = \{\tau \subseteq P \cap B(v',\rho) \mid \pi_{\mathbf{T}_{v'}\mathcal{M}}(\tau) \in \mathrm{Star}_{P,\mathcal{M}}(v',\rho)\}$

$$\mathrm{TangDel}_{\mathcal{M}}(P,\rho) = \bigcup_{v \in P} \mathrm{Prestar}_{P,\mathcal{M}}(v,\rho)$$

Simplices are ρ-small

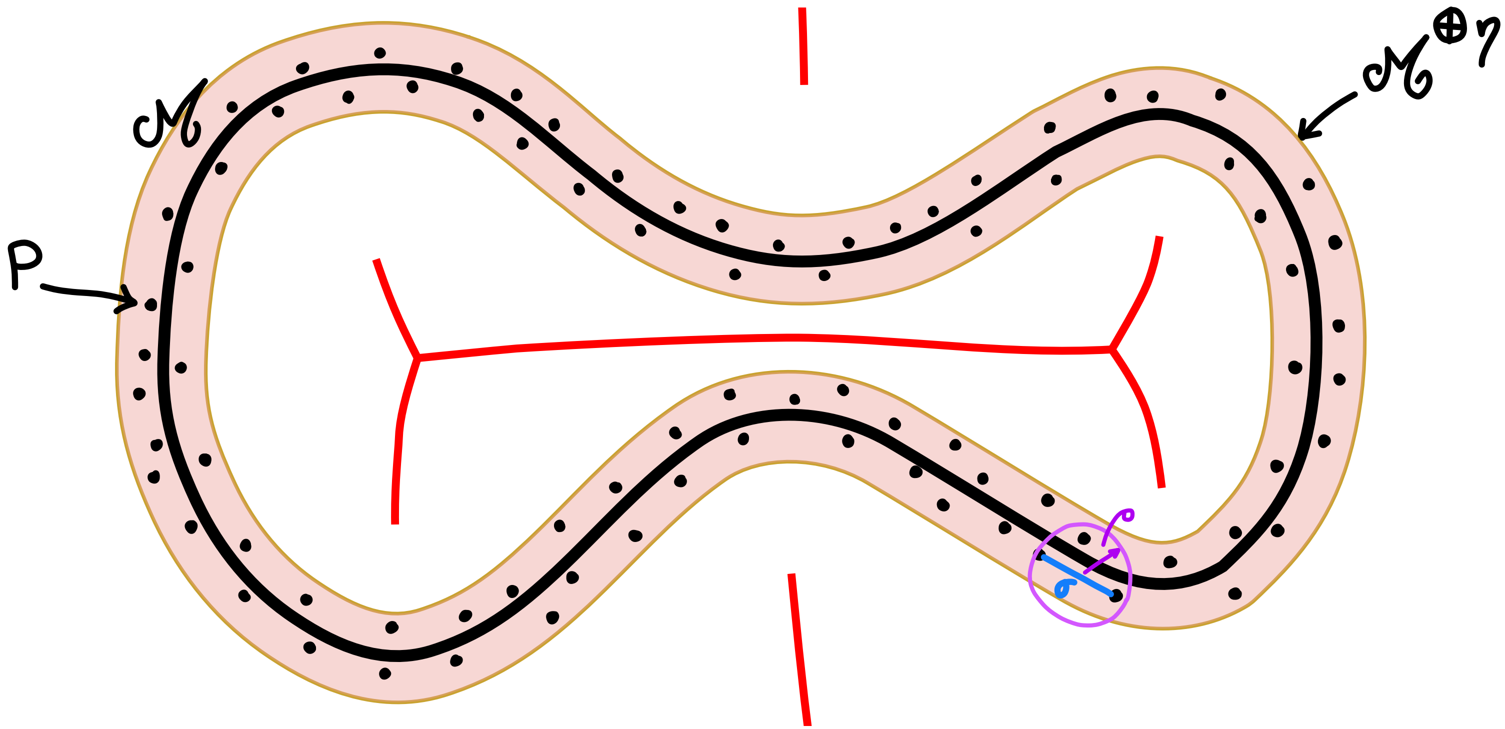
Construction involves computing $|P|$ small d-dimensional Delaunay complexes

Reach and projection



- Medial axis of $\mathcal{M}$ = set of points with at least 2 closest points onto $\mathcal{M}$
- $\text{Reach}(\mathcal{M}) = d(\mathcal{M}, \text{Medial axis of } \mathcal{M})$

Reach and projection



- Medial axis of $\mathcal{M}$ = set of points with at least 2 closest points onto $\mathcal{M}$
- $\text{Reach}(\mathcal{M}) = d(\mathcal{M}, \text{Medial axis of } \mathcal{M})$

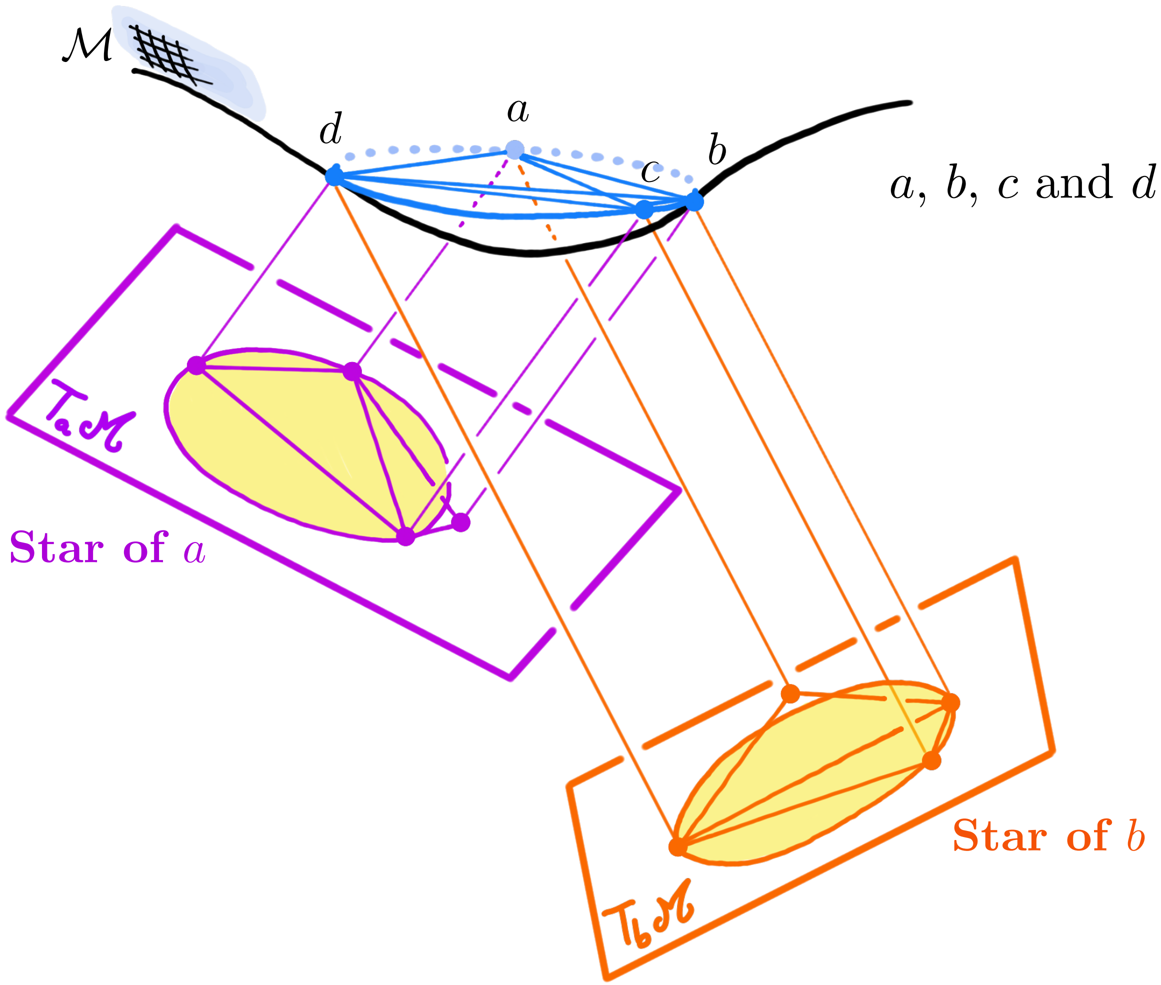
We assume $P \subseteq \mathcal{M}^{\oplus\eta}$ and scale ρ such that $\eta + \rho < \text{Reach}(\mathcal{M})$

$$\forall \sigma \subseteq P \;\; \rho\text{-small}, \quad \pi_{\mathcal{M}} : \text{Conv}(\sigma) \to \mathcal{M} \text{ well-defined}$$

Unweighted tangentiel Delaunay complex

$$\mathrm{TangDel}_{\mathcal{M}}(P,\rho) = \bigcup_{v \in P} \mathrm{Prestar}_{P,\mathcal{M}}(v,\rho)$$

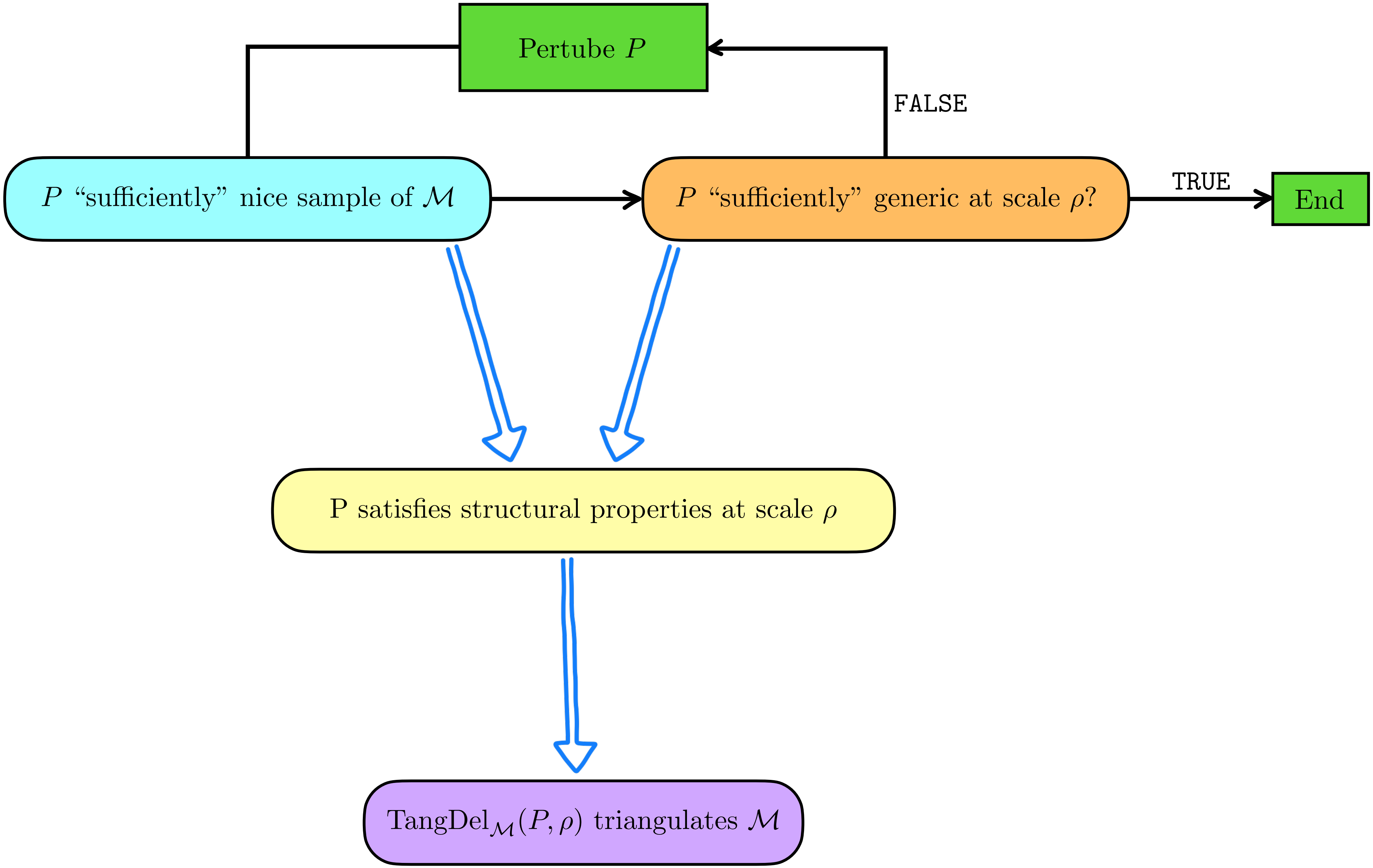
⚠ **Not necessarily a triangulation of the submanifold!**



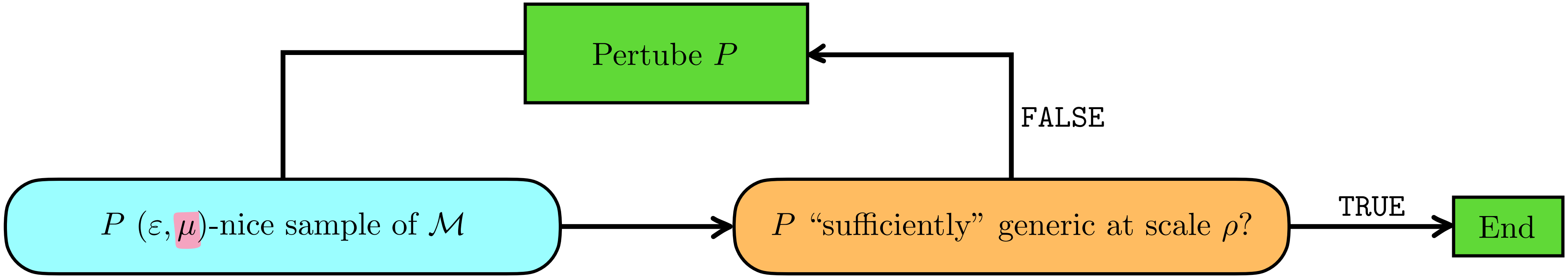
a, b, c and d almost cocircular

Prestar of a and Prestar of b do not agree on triangles abc, abd, acd, bcd!

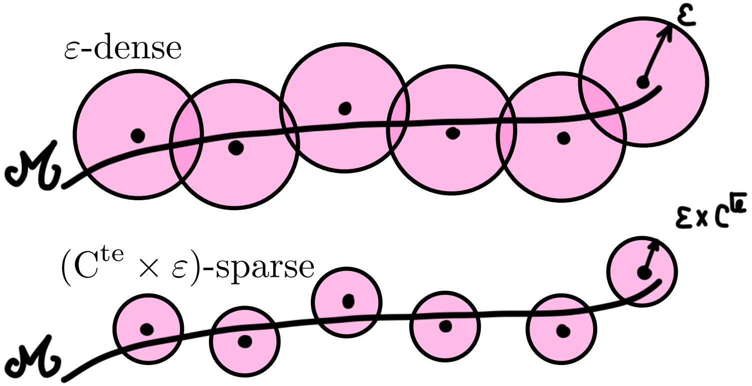
How to ensure that triangulation?



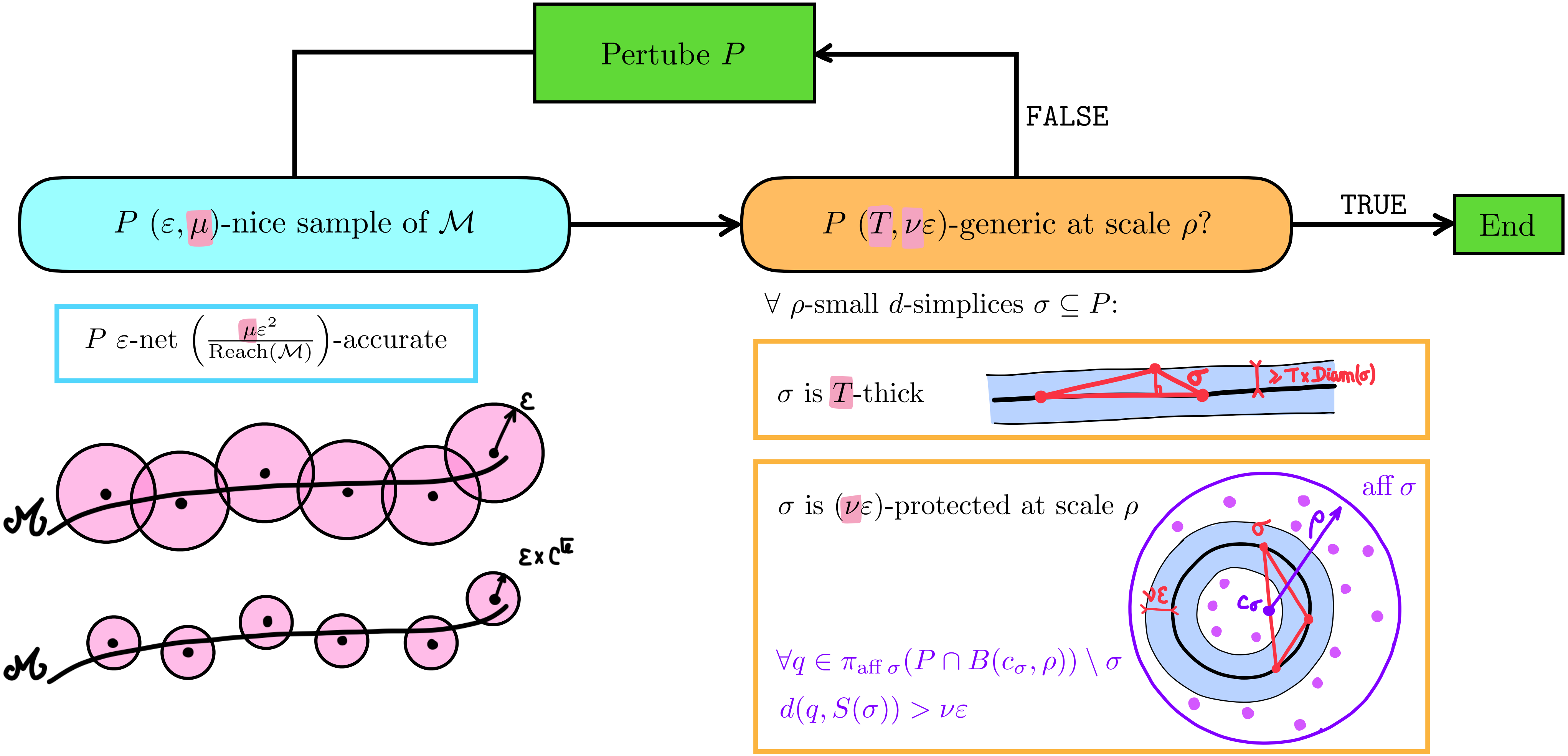
How to ensure that triangulation?



P ε-net $\left(\frac{\mu\varepsilon^2}{\mathrm{Reach}(\mathcal{M})}\right)$-accurate



How to ensure that triangulation?



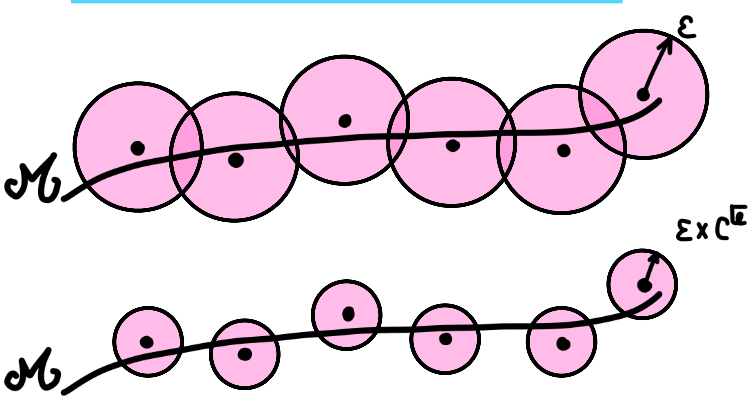
How to ensure that triangulation?

Pertube P

FALSE

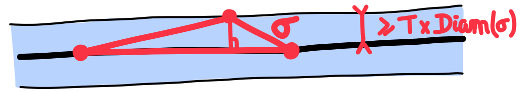
P (ε, μ)-nice sample of $\mathcal{M}$ → P $(T, \nu\varepsilon)$-generic at scale ρ? → TRUE → End

P ε-net $\left(\frac{\mu\varepsilon^2}{\text{Reach}(\mathcal{M})}\right)$-accurate

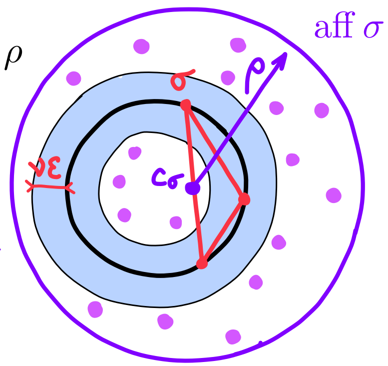


$\forall$ ρ-small d-simplices $\sigma \subseteq P$:

σ is T-thick



σ is $(\nu\varepsilon)$-protected at scale ρ



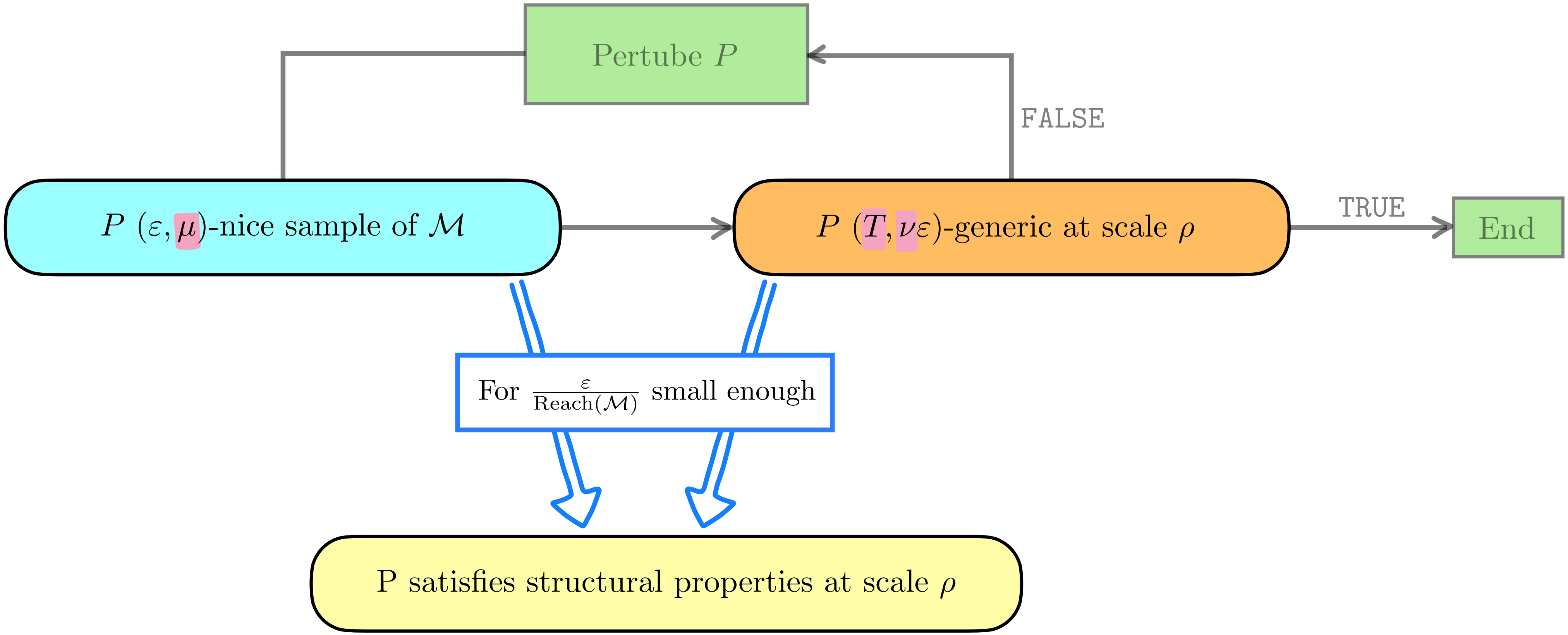
$\forall q \in \pi_{\text{aff}\,\sigma}(P \cap B(c_\sigma, \rho)) \setminus \sigma$

$d(q, S(\sigma)) > \nu\varepsilon$

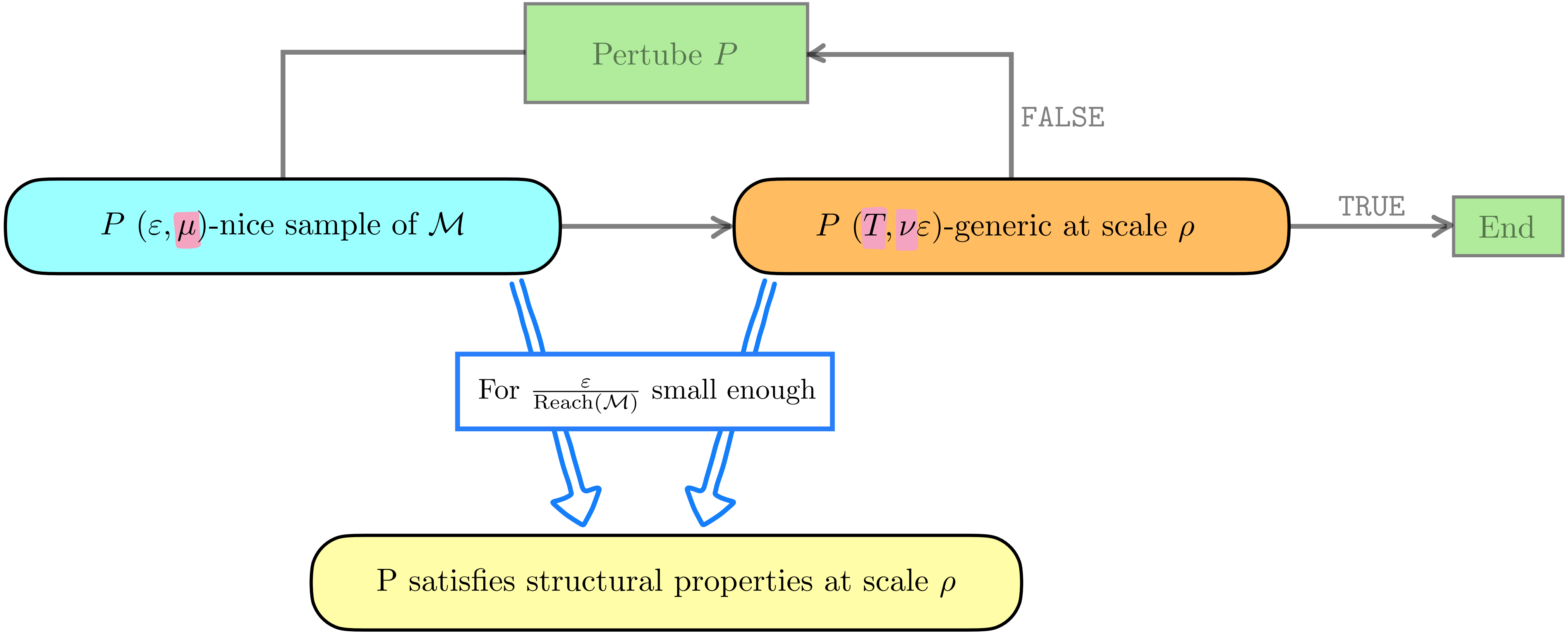
Set scale parameter: $\rho = \lambda\varepsilon$ with $\lambda \geq 6$

Lovácz Local Lemma $\implies$ $\exists$ μ, T, ν, C such that for $\frac{\varepsilon}{\text{Reach}(\mathcal{M})} < C$, the algorithm terminates

How to ensure that triangulation?



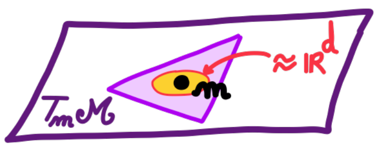
How to ensure that triangulation?



$\forall m \in \mathcal{M}$

$\pi_{\mathbf{T}_m\mathcal{M}}$ injective on $P \cap B(m, \rho)$

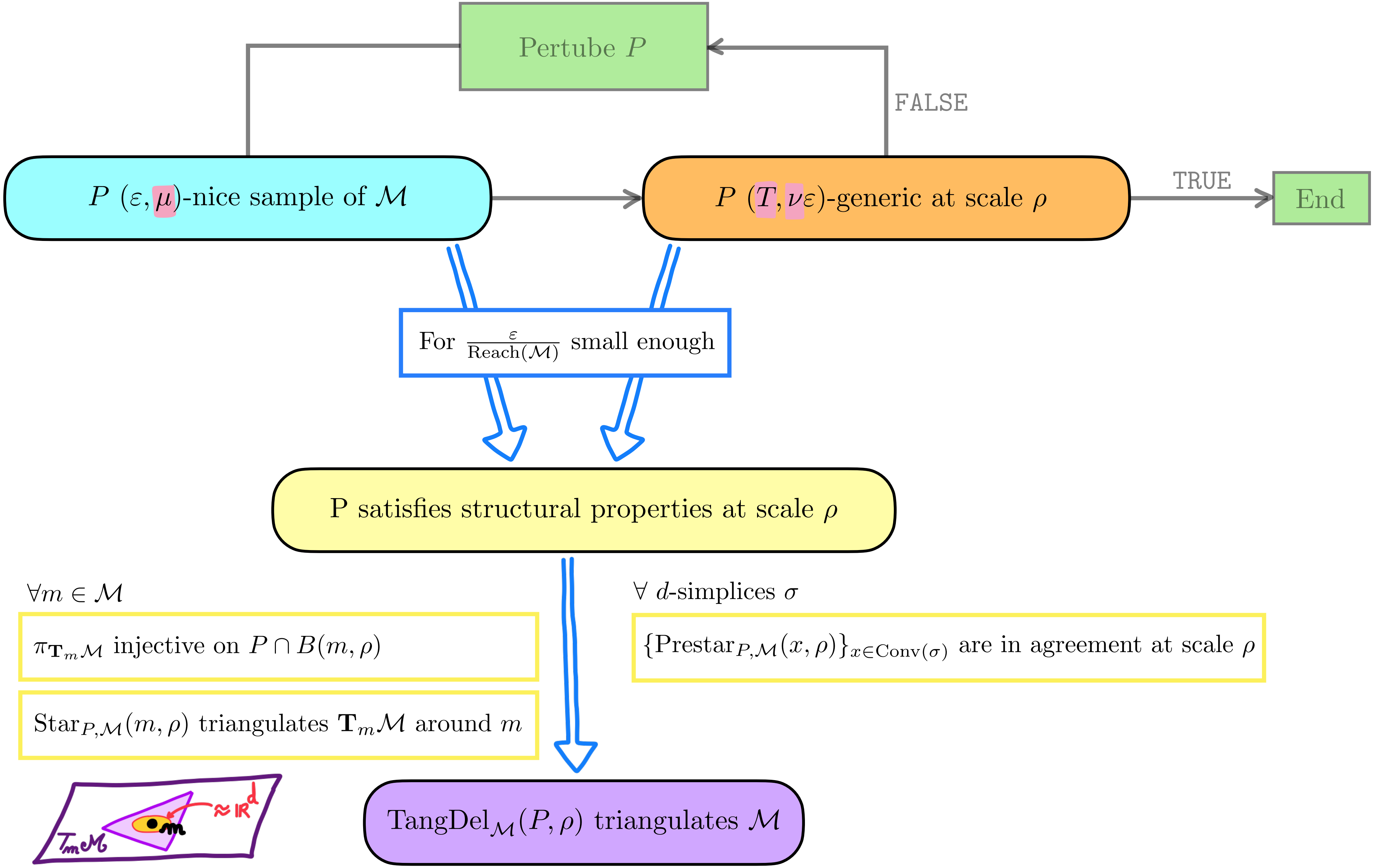
$\mathrm{Star}_{P,\mathcal{M}}(m, \rho)$ triangulates $\mathbf{T}_m\mathcal{M}$ around m



$\forall$ d-simplices σ

$\{\mathrm{Prestar}_{P,\mathcal{M}}(x, \rho)\}_{x \in \mathrm{Conv}(\sigma)}$ are in agreement at scale ρ

How to ensure that triangulation?

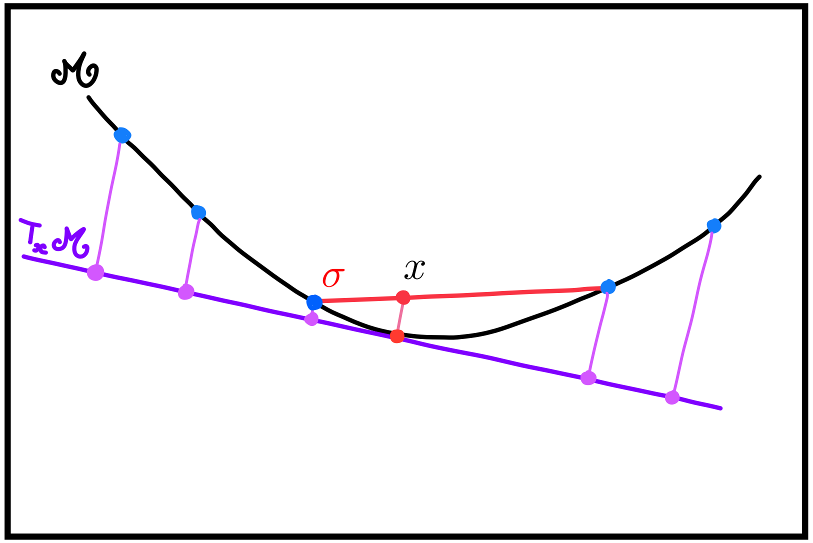


Prestars in agreement

Prestars of σ are *in agreement* at scale ρ if

$$\forall x, y \in \mathrm{Conv}(\sigma)$$

$$\sigma \in \mathrm{Prestar}_{P,\mathcal{M}}(x, \rho) \iff \sigma \in \mathrm{Prestar}_{P,\mathcal{M}}(y, \rho)$$

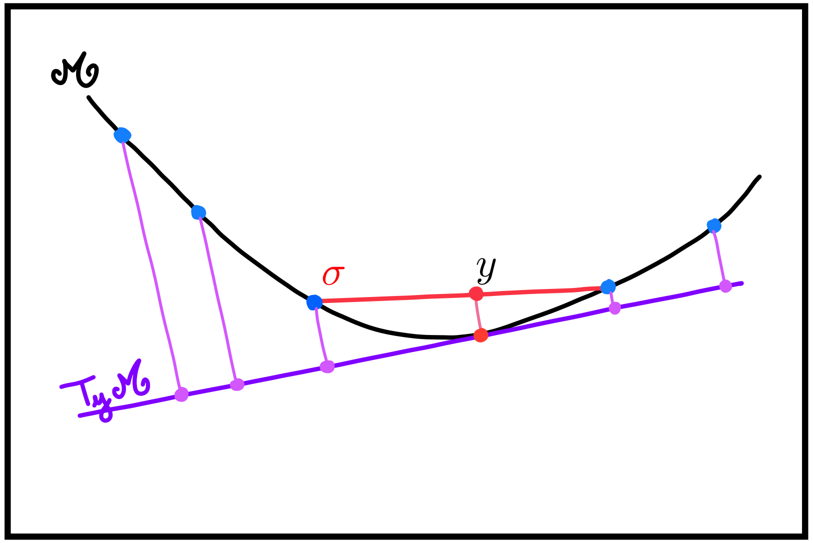


Prestars in agreement

Prestars of σ are *in agreement* at scale ρ if

$$\forall x, y \in \mathrm{Conv}(\sigma)$$

$$\sigma \in \mathrm{Prestar}_{P,\mathcal{M}}(x,\rho) \iff \sigma \in \mathrm{Prestar}_{P,\mathcal{M}}(y,\rho)$$

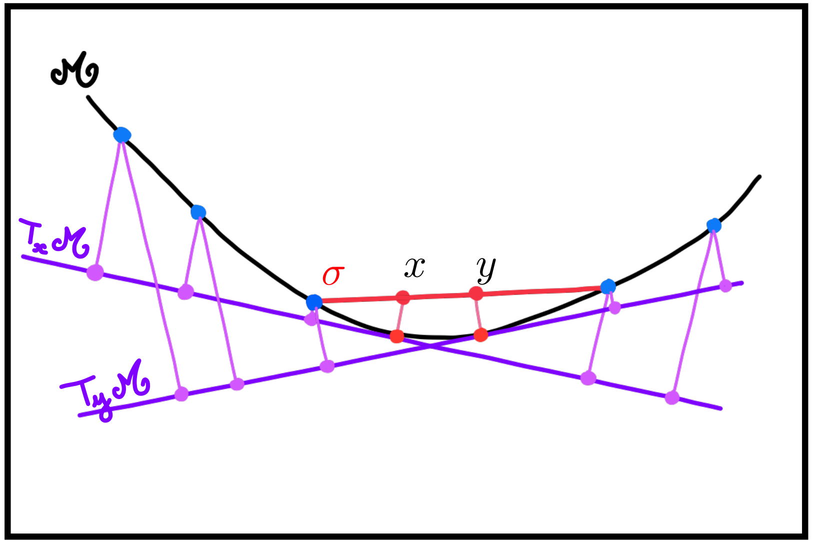


Prestars in agreement

Prestars of σ are *in agreement* at scale ρ if

$$\forall x, y \in \text{Conv}(\sigma)$$

$$\sigma \in \text{Prestar}_{P,\mathcal{M}}(x, \rho) \iff \sigma \in \text{Prestar}_{P,\mathcal{M}}(y, \rho)$$



σ projects onto a Delaunay simplex either in all "nearby" tangent planes or in none of them.

We can work in the tangent plane that is most convenient for us!

Triangulation of the manifold

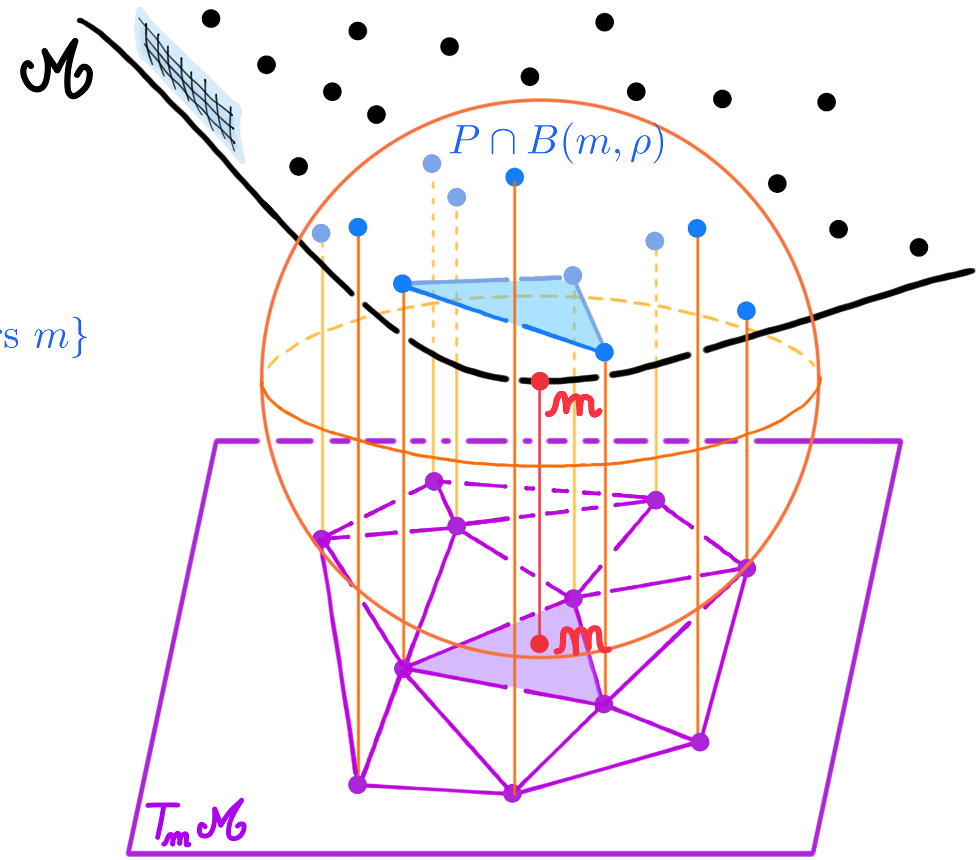
$\mathrm{Prestar}_{P,\mathcal{M}}(m,\rho)$

$\shortparallel$

$\{\sigma \in \mathrm{TangDel}_{\mathcal{M}}(P,\rho) \mid \pi_{\mathcal{M}}(\sigma) \text{ covers } m\}$

$\Uparrow$

$\mathrm{Star}_{P,\mathcal{M}}(m,\rho)$

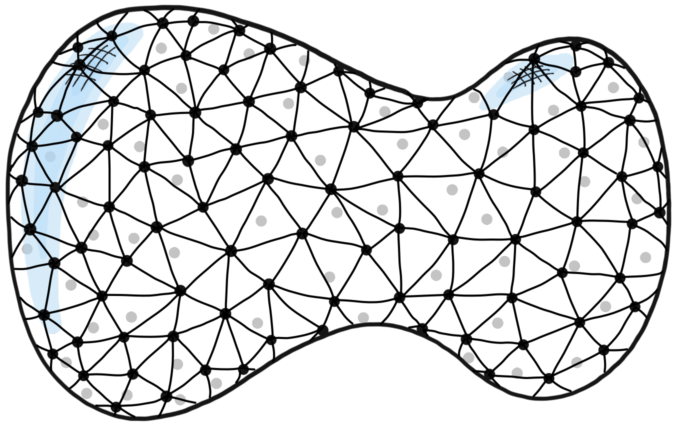


- We prove that $\mathcal{D} = |\,\mathrm{TangDel}_{\mathcal{M}}(P,\rho)|$ d-manifold and $\pi_{\mathcal{M}} : \mathcal{D} \to \mathcal{M}$ injective.
- Domain invariance theorem $\implies$ $\pi_{\mathcal{M}} : \mathcal{D} \to \mathcal{M}$ open

$\pi_{\mathcal{M}} : |\,\mathrm{TangDel}_{\mathcal{M}}(P,\rho)| \to \mathcal{M}$ homeomorphism

Unweighted tangential Delaunay complex

$\mathrm{TangDel}_{\mathcal{M}}(P, \rho)$



Geometric characterization of elements ✓

P samples $\mathcal{M}$ “sufficently” and “sufficiently“ generic

⇓

Triangulation of $\mathcal{M}$

Variational characterization ?

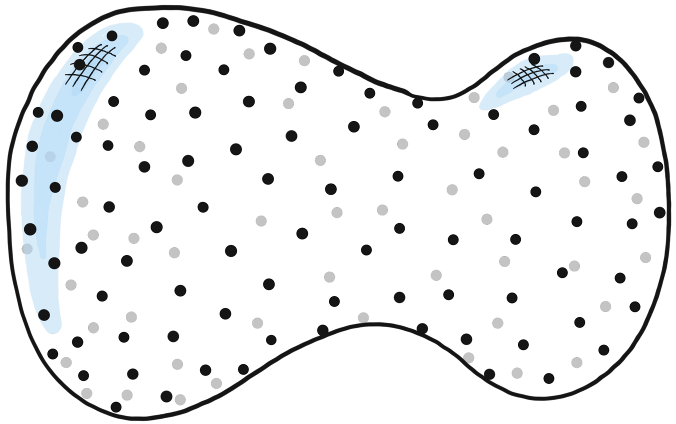
P samples $\mathcal{M}$ “sufficently” and “sufficiently“ generic

⇓

Minimizes Delaunay energy

Finding a triangulation by minimization

P: a finite sample of a smooth d-submanifold $\mathcal{M}$ of $\mathbb{R}^N$



Find a set of d-simplices T that

- minimizes $E_{\mathrm{del}}(T) = \sum_{\sigma\ d\text{-simplex of } T} \omega_{\mathrm{del}}(\sigma)$
- subject to: "T is a mesh with vertex set P that triangulates $\mathcal{M}$"

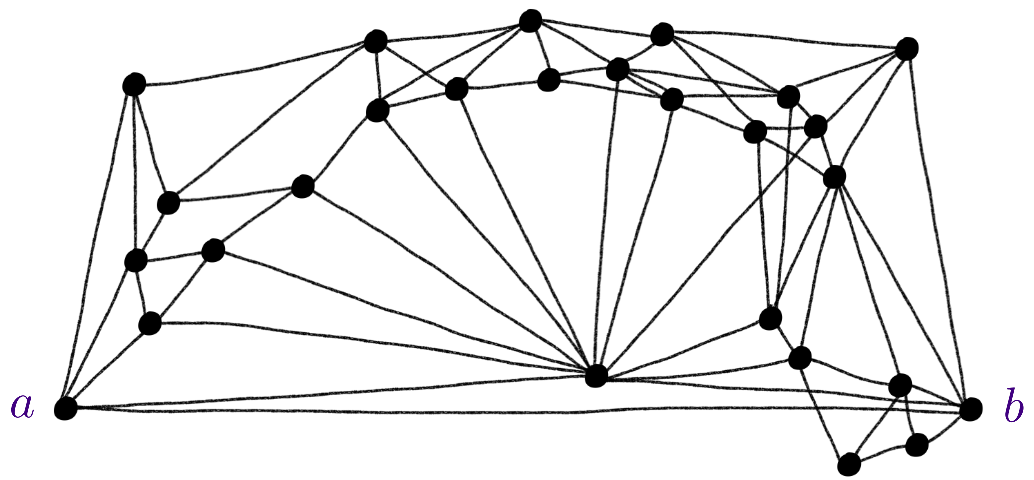
Why using Delaunay weights?

How to transform this into a convex problem?

Finding a path by minimization

a graph $G = (V, E)$

$\omega(e) =$ weight of e



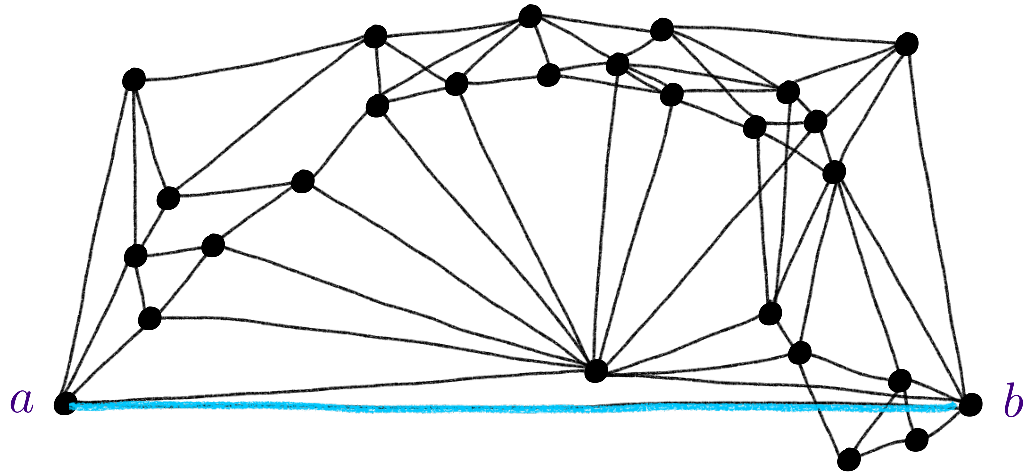
Find a path γ that

- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)$
- subject to: γ connects a to b

Finding a path by minimization

a graph $G = (V, E)$

$\omega(e) = \text{length}(e)$



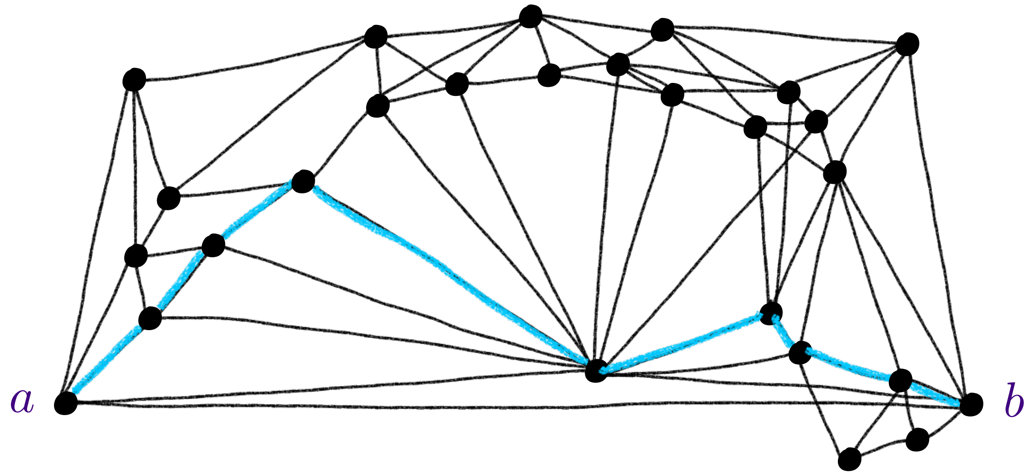
Find a path γ that
- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)$
- subject to: γ connects a to b

The shortest path is ab (assuming $ab \in G$).

Finding a path by minimization

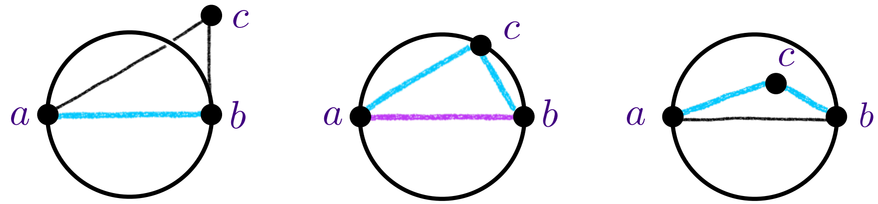
a graph $G = (V, E)$

$\omega(e) = \text{length}(e)^2$



Find a path γ that
- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)$
- subject to: γ connects a to b

Pythagorean theorem

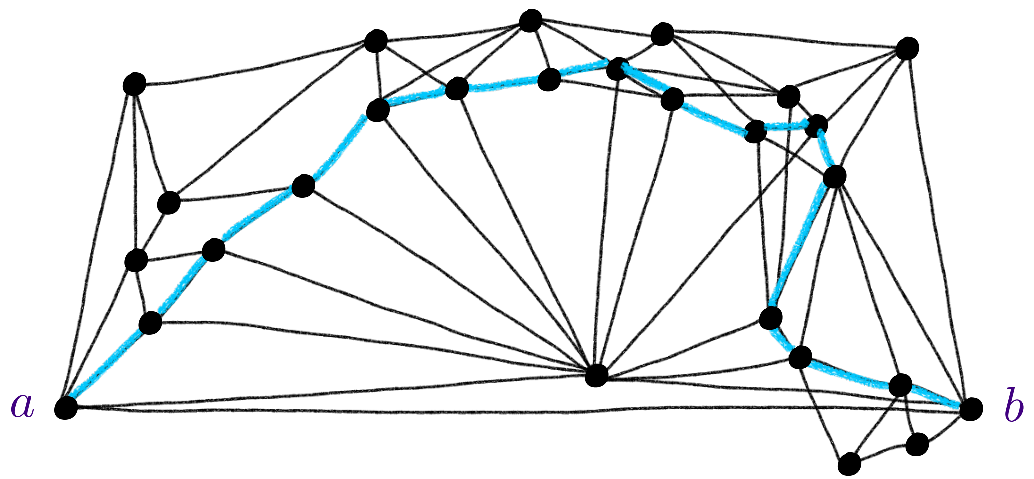


Finding a path by minimization

a graph $G = (V, E)$

$\omega(e) = \text{length}(e)^3$

The Delaunay weight



Find a path γ that

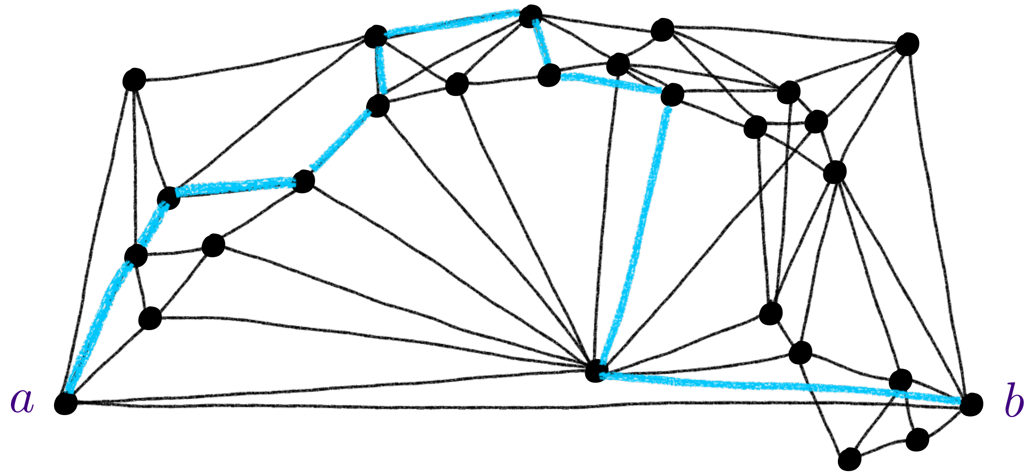
- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)$
- subject to: γ connects a to b

The shortest path starts going through “denser” parts of the point cloud!

Finding a path by minimization

a graph $G = (V, E)$

$\omega(e)$ = weight of e



Find a path γ that

- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)$
- subject to: γ connects a to b

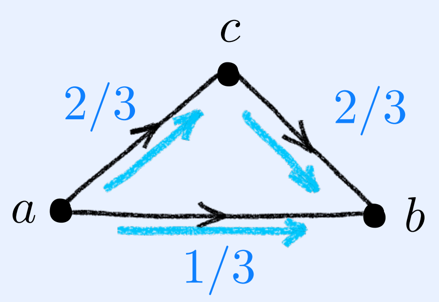
Dijkstra's algorithm solves the problem in $O(|E| + |V| \log |V|)$.

Enlarging the search space

the set of paths ⟶ the set of 1-chains with coefficients in $\mathbb{R}$

finite — vector space

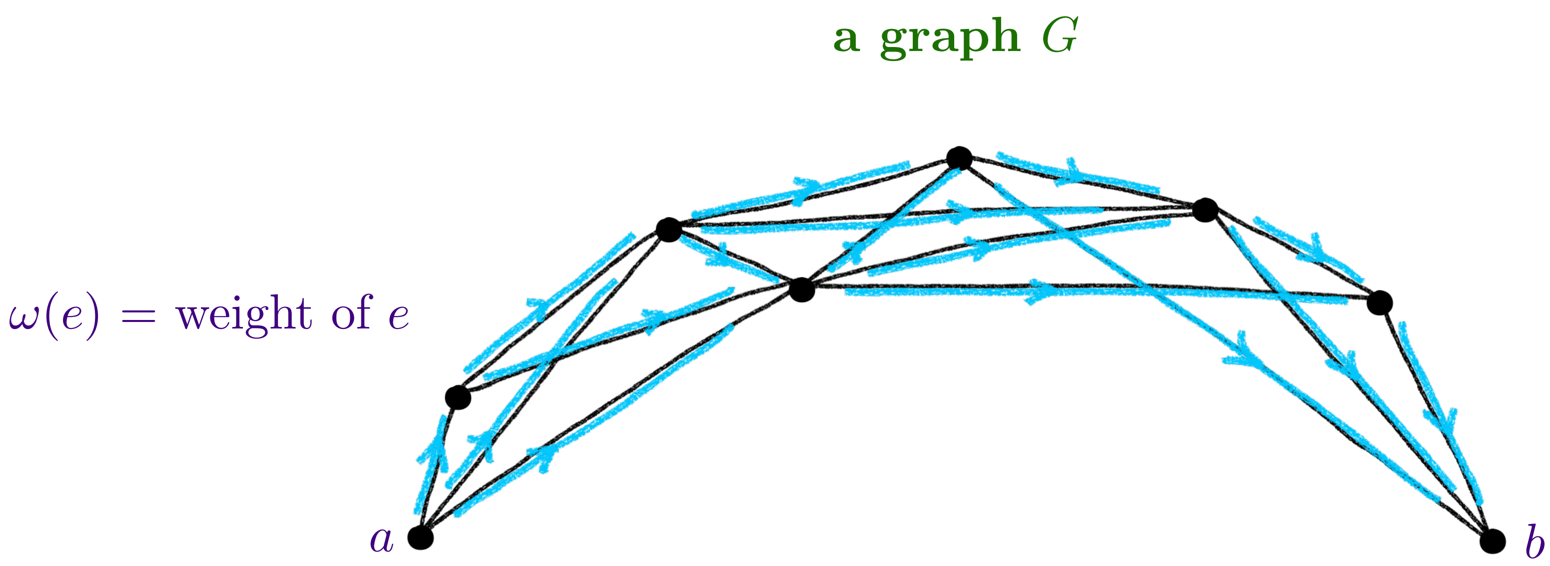
- give each edge e an arbitrary orientation:
- 1-chain $$\gamma = \sum_{e \text{ edge}} \gamma(e) e$$
  vector — coordinate — element of the basis
- boundary operator ∂: linear operator such that $\partial e = v_{\text{end}}(e) - v_{\text{start}}(e)$



$$\gamma = \frac{2}{3}[ac] + \frac{2}{3}[cb] + \frac{1}{3}[ab]$$

$$\partial\gamma = \frac{2}{3}(c-a) + \frac{2}{3}(b-c) + \frac{1}{3}(b-a) = b - a$$

Reformulating minimization problem



L_2 norm

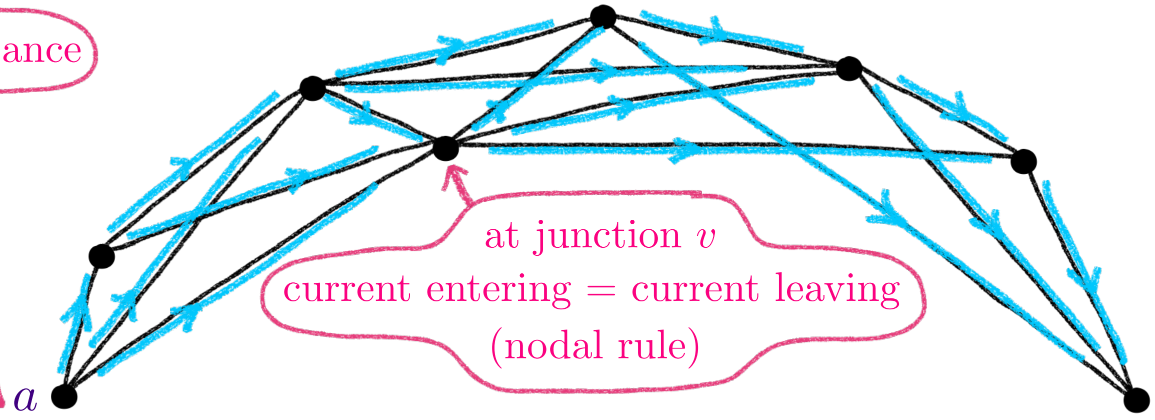
Find a 1-chain γ that

- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)\, \gamma(e)^2$
- subject to: $\partial\gamma = b - a$

If $a = b$, the solution is a harmonic form γ and can be computed using $W^{\frac{1}{2}}\gamma \in \ker(W^{\frac{1}{2}}\partial_{d+1}\partial_{d+1}^t W^{\frac{1}{2}} + W^{-\frac{1}{2}}\partial_d^t\partial_d W^{-\frac{1}{2}})$.

The solution spreads everywhere!

Physical interpretation



current — Joule heating — L_2 norm

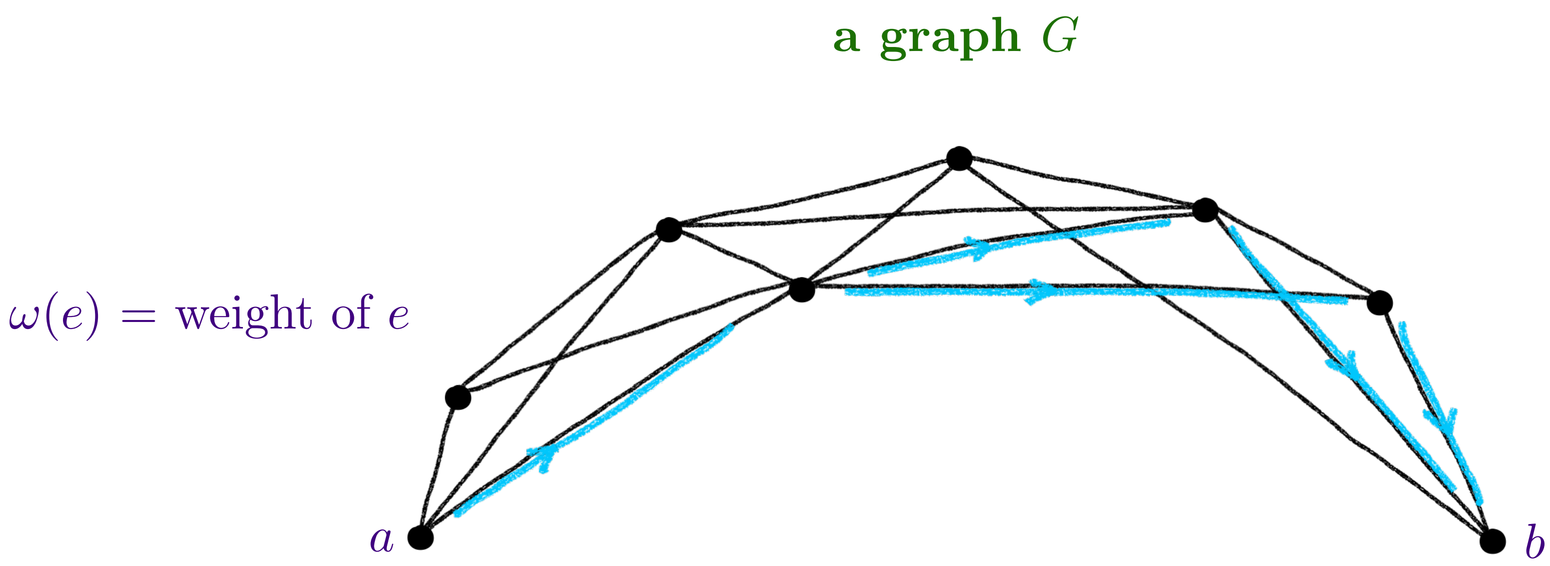
Find a 1-chain γ that

- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)\, \gamma(e)^2$ ← mesh rule
- subject to: $\partial\gamma = b - a$ ← nodal rule

If $a = b$, the solution is a harmonic form γ and can be computed using $W^{\frac{1}{2}}\gamma \in \ker(W^{\frac{1}{2}}\partial_{d+1}\partial_{d+1}^t W^{\frac{1}{2}} + W^{-\frac{1}{2}}\partial_d^t\partial_d W^{-\frac{1}{2}})$.

The solution spreads everywhere!

Reformulating minimization problem



L_1 norm

Find a 1-chain γ that

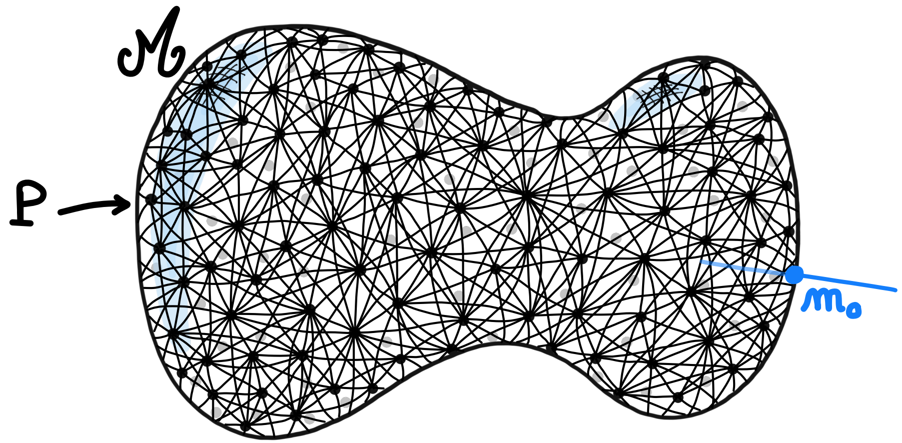
- minimizes $E(\gamma) = \sum_{e \text{ edge of } \gamma} \omega(e)\,|\gamma(e)|$
- subject to: $\partial\gamma = b - a$

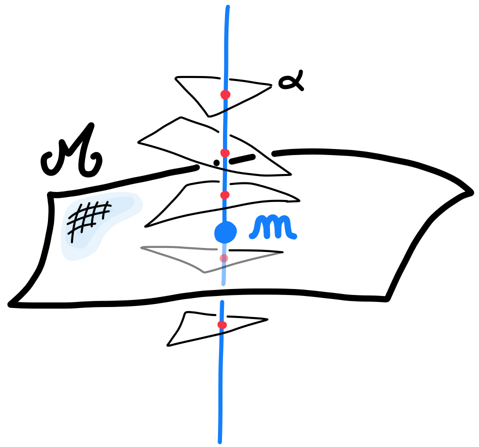
Convex problem whose solution can be computed by linear programming (using slack variables)

The solution is expected to be sparse! Not necessarily a path!

Reformulating minimization problem

K: a simplicial complex with vertex set P





$$\text{load}_m(\gamma) = \sum_{\alpha} \gamma(\alpha) \mathbf{1}_{\pi_{\mathcal{M}}(\text{Conv}(\alpha))}(m)$$

Find a d-chain γ of K that

- minimizes $E_{\text{del}}(\gamma) = \sum_{\sigma\ d\text{-simplex of } K} \omega_{\text{del}}(\sigma)\,|\gamma(\sigma)|$
- subject to: $\partial\gamma = 0$ ← cycle

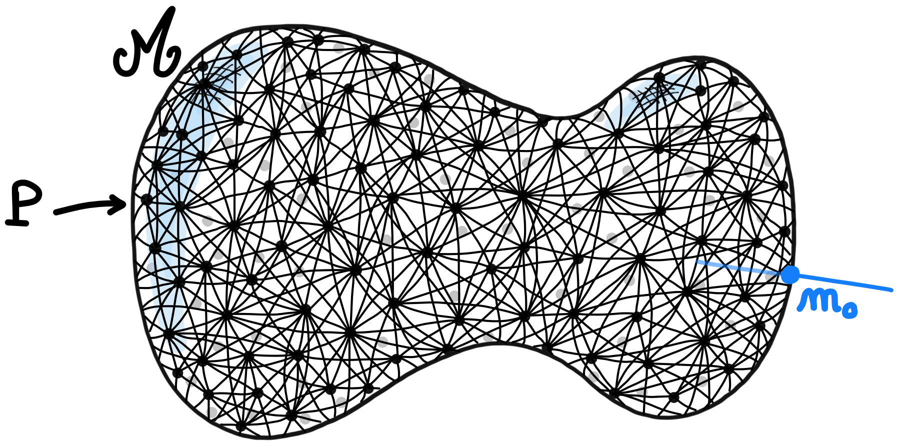
  $\text{load}_{m_0}(\gamma) = 1$ ← "normalization"

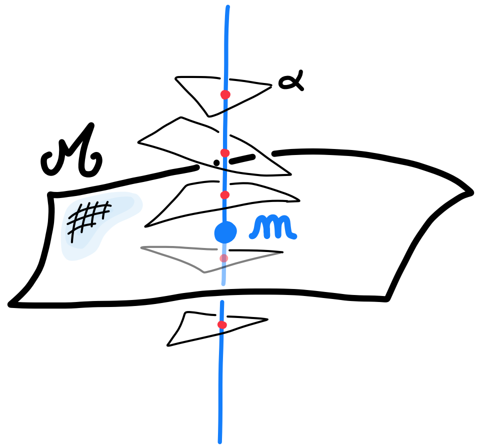
⇒ Least-norm problem whose constraint functions ∂ and load_{m_0} are linear

Convex optimization problem

Our result

K: **a simplicial complex with vertex set** P





$$\text{load}_m(\gamma) \triangleq \sum_{\alpha} \gamma(\alpha) \mathbf{1}_{\pi_{\mathcal{M}}(\text{Conv}(\alpha))}(m)$$

Find a d-chain γ of K that

- minimizes $E_{\text{del}}(\gamma) = \sum_{\sigma \text{ } d\text{-simplex of } K} \omega_{\text{del}}(\sigma)\, |\gamma(\sigma)|$
- subject to: $\partial\gamma = 0$

  $\text{load}_{m_0}(\gamma) = 1$

Theorem. *There exists a constant C such that if P is (ε, μ)-nice sample of $\mathcal{M}$ $(T, \nu\varepsilon)$-generic at scale ρ and* $\text{TangDel}_{\mathcal{M}}(P, \rho) \subseteq K \subseteq \text{Cech}(P, \rho)$,

then for $\frac{\varepsilon}{\text{Reach}(\mathcal{M})} < C$, the solution is unique and defines a triangulation of $\mathcal{M}$ which is $\text{TangDel}_{\mathcal{M}}(P, \rho)$.

Idea of the proof

Find a d-chain γ of K that

- minimizes $E_{\text{del}}(\gamma) = \sum_{\sigma \text{ } d\text{-simplex of } K} \omega_{\text{del}}(\sigma)\, |\gamma(\sigma)|$
- subject to: $\partial\gamma = 0$

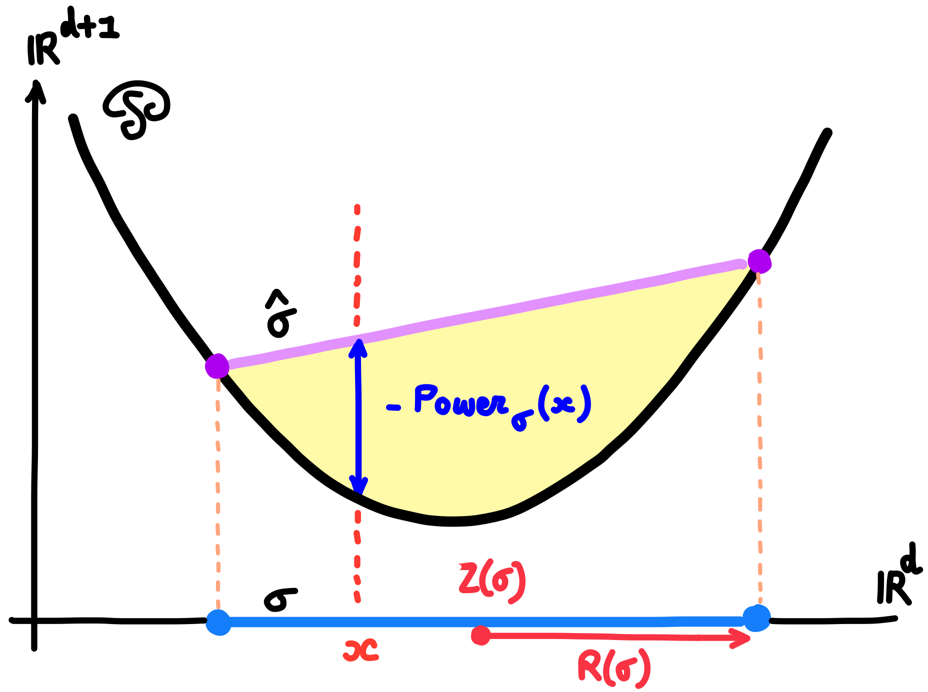
  $\text{load}_{m_0}(\gamma) = 1$

$\omega_{\text{del}}(\sigma) \triangleq$ volume between $\hat{\sigma}$ and $\mathcal{P}$

$$= \frac{1}{(d+1)(d+2)} \text{vol}(\sigma) \sum_{e \text{ edge of } \sigma} \text{length}(e)^2$$

(intrinsec expression)

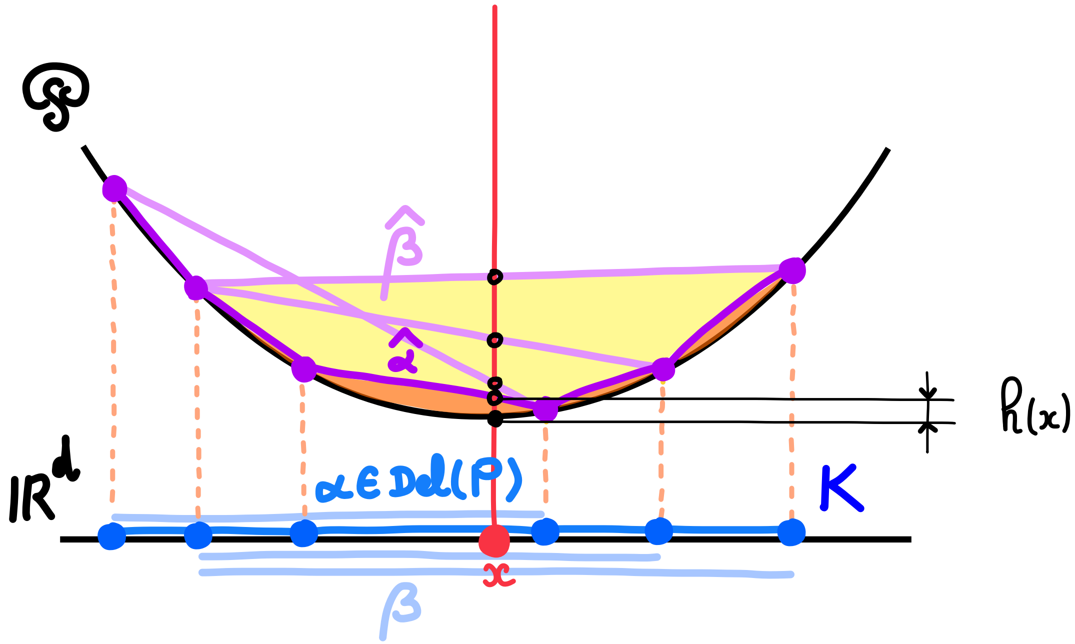
$$= \int_{x \in \text{Conv}(\sigma)} -\text{Power}_\sigma(x)\, dx$$



where $\text{Power}_\sigma(x)$ = power distance of x to smallest circumsphere of σ $= \|x - Z(\sigma)\|^2 - R(\sigma)^2$

Idea of the proof

Euclidean case



- γ_{del} = chain defined by $\mathrm{Del}(P)$

$$= \text{ iff } \gamma = \gamma_{\mathrm{del}}$$

$$E_{\mathrm{del}}(\gamma_{\mathrm{del}}) = \sum_{\sigma} \omega_{\min}(\sigma)|\gamma_{\min}(\sigma)| \leq \sum_{\sigma} \omega_{\min}(\sigma)|\gamma(\sigma)| \leq \sum_{\sigma} \omega_{\mathrm{del}}(\sigma)|\gamma(\sigma)| = E_{\mathrm{del}}(\gamma)$$

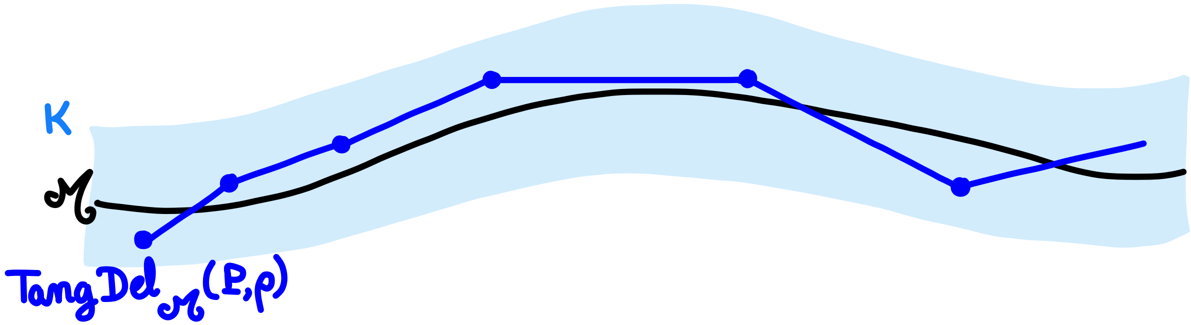
volume between $\mathrm{LowConv}(\hat{P})$ and $\mathcal{P}$

volume between $\mathrm{LowConv}(\hat{P})$ above σ and $\mathcal{P}$ $= \int_{x \in \mathrm{Conv}(\sigma)} h(x)\, dx$

$$h(x) = \min_{\substack{\sigma\ d\text{-simplex} \\ x \in \mathrm{Conv}(\sigma)}} (-\mathrm{Power}_{\sigma}(x))$$

Idea of the proof

Manifold case



Idea of the proof

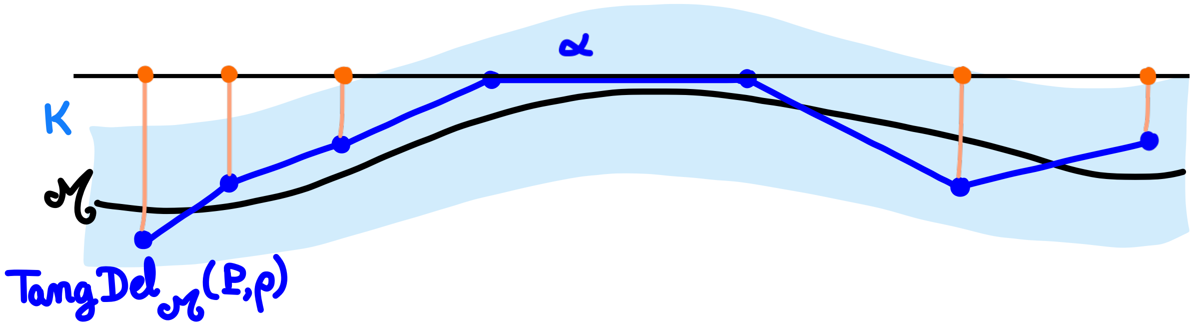
Manifold case

P "sufficiently" generic at scale ρ

$\alpha \in \mathrm{TangDel}_{\mathcal{M}}(P, \rho)$

$\Updownarrow$

$\alpha \in \mathrm{Del}(\pi_{\mathrm{aff}\,\alpha}(P \cap B(c_\alpha, \rho)))$



Idea of the proof

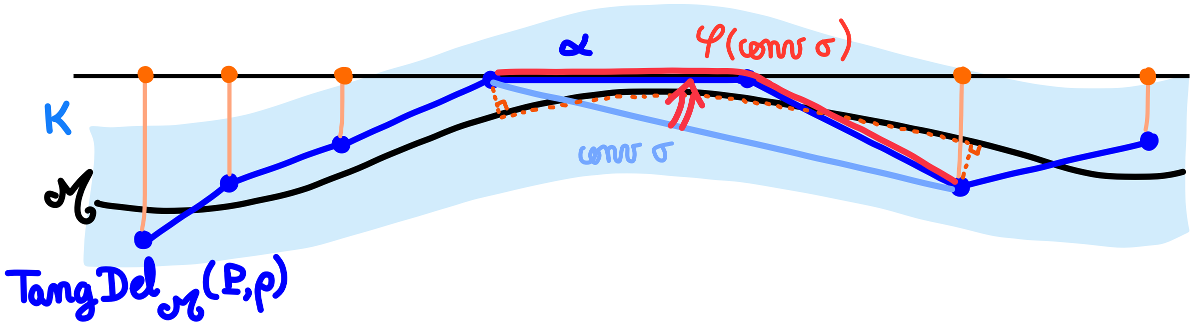
Manifold case

P "sufficiently" generic at scale ρ

$\alpha \in \text{TangDel}_{\mathcal{M}}(P, \rho)$

⇕

$\alpha \in \text{Del}(\pi_{\text{aff}\,\alpha}(P \cap B(c_\alpha, \rho)))$



Idea of the proof

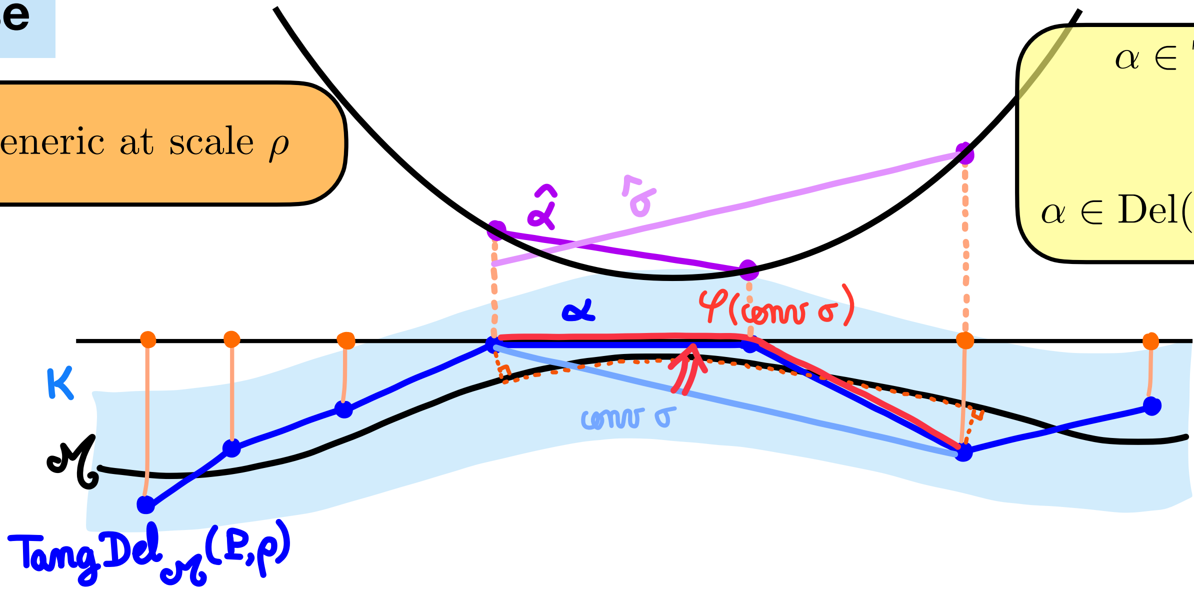
Manifold case

P "sufficiently" generic at scale ρ

$\alpha \in \mathrm{TangDel}_{\mathcal{M}}(P, \rho)$

$\Updownarrow$

$\alpha \in \mathrm{Del}(\pi_{\mathrm{aff}\,\alpha}(P \cap B(c_\alpha, \rho)))$



- γ_{del} = chain defined by $\mathrm{TangDel}_{\mathcal{M}}(P, \rho)$

$=$ iff $\gamma = \gamma_{\mathrm{del}}$

$$E_{\mathrm{del}}(\gamma_{\mathrm{del}}) = \sum_\sigma \omega_{\min}(\sigma)|\gamma_{\min}(\sigma)| \leq \sum_\sigma \omega_{\min}(\sigma)|\gamma(\sigma)| \leq \sum_\sigma \omega_{\mathrm{del}}(\sigma)|\gamma(\sigma)| = E_{\mathrm{del}}(\gamma)$$

volume between $\mathrm{LowConv}(\hat{P})$ and $\mathcal{P}$

$$\int_{x \in \varphi(\mathrm{Conv}(\sigma))} h(x)\,dx$$

$$h(x) = \min_{\substack{\sigma\ d\text{-simplex} \\ y \in \mathrm{Conv}(\sigma) \\ \pi_{\mathcal{M}}(y) = \pi_{\mathcal{M}}(x)}} (-\mathrm{Power}_\sigma(y))$$

Experiments

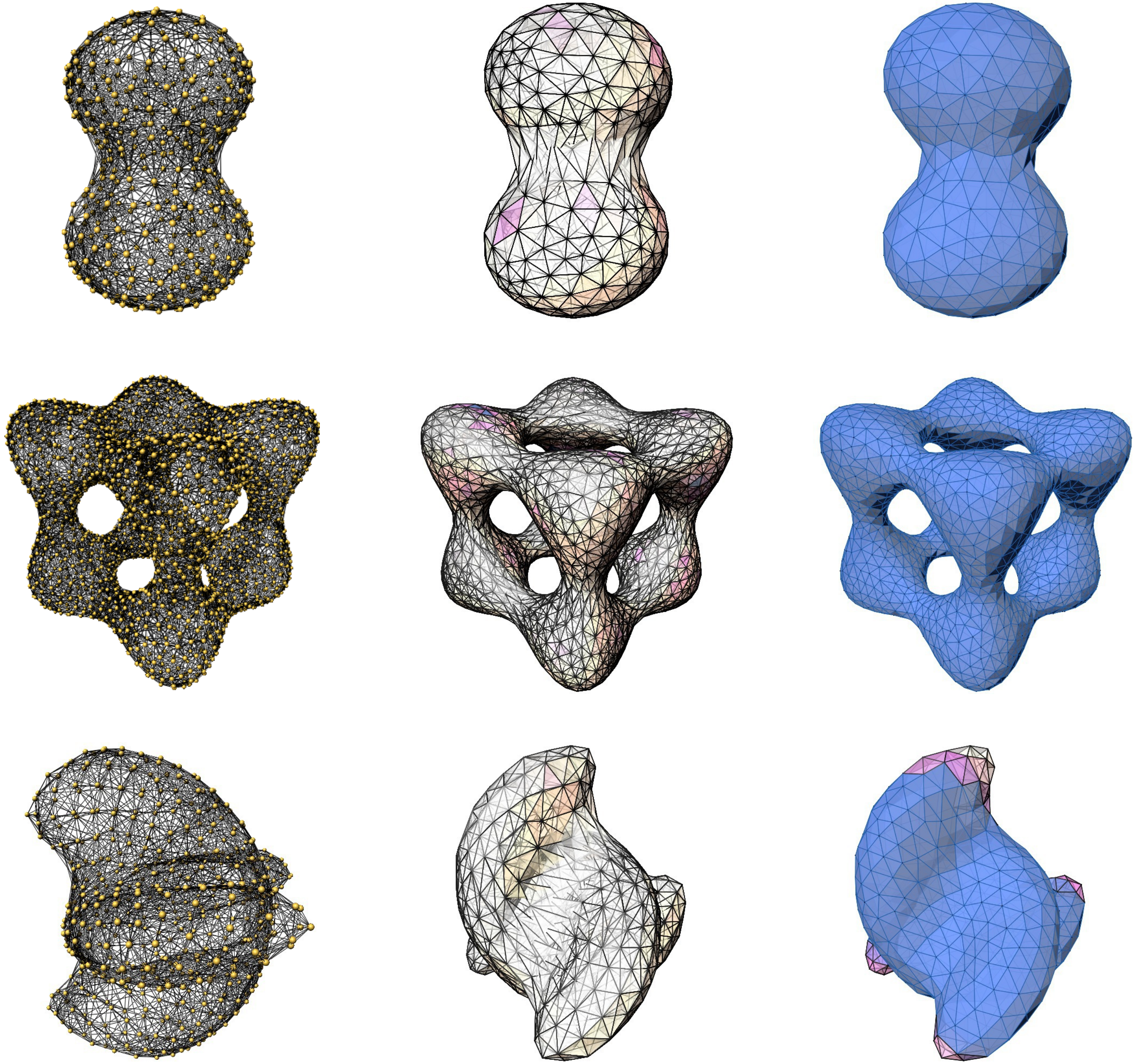
Harmonic form
(L2-minimization)

Solution
(L1-minimization)

Rips complex
(382 vertices, $E(\text{deg}) = 17.1$)

Rips complex
(2000 vertices, $E(\text{deg}) = 17.1$)

Rips complex
(542 vertices, $E(\text{deg}) = 17.5$)



Conclusion

Two papers in preparation

Future work

- **Algorithmic aspects**
- **Anisotropic energy**

Thank you for your attention!